

Asymptotic behavior of the Eden model with positively homogeneous edge weights

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Abstract

Let $d \in \mathbb{N}$, $\alpha \in \mathbb{R}$, and let $f : \mathbb{R}^d \setminus \{0\} \rightarrow (0, \infty)$ be locally Lipschitz and positively homogeneous of degree α (e.g. f could be the α th power of a norm on \mathbb{R}^d). We study a generalization of the Eden model on \mathbb{Z}^d wherein the next edge added to the cluster is chosen from the set of all edges incident to the current cluster with probability proportional to the value of f at the midpoint of this edge, rather than uniformly. This model is equivalent to a variant of first passage percolation where the edge passage times are independent exponential random variables with parameters given by the value of f at the midpoint of the edge.

We prove that the f -weighted Eden model clusters have an a.s. deterministic limit shape if $\alpha < 1$, which is an explicit functional of f and the limit shape of the standard Eden model, and estimate the rate of convergence to this limit shape. We also prove that if $\alpha > 1$, then there is a norm ν on \mathbb{R}^d (depending on α) such that if we set $f(z) = \nu(z)^\alpha$, then the f -weighted Eden model clusters are a.s. contained in a Euclidean cone with opening angle $< \pi$ for all time. We further show that there does *not* exist a norm on \mathbb{R}^d for which this latter statement holds for all $\alpha > 1$; and that there is no choice of function f for which the above statement holds with $\alpha = 1$.

Our basic approach is to compare the local behavior of the f -weighted first passage percolation to that of unweighted first passage percolation with iid exponential edge weights (which is equivalent to the unweighted Eden model).

We include a list of open problems and several computer simulations.

Contents

| | | |
|----------|------------------------------------------------------------|-----------|
| 1 | Introduction | 2 |
| 1.1 | Overview | 2 |
| 1.2 | Basic notations | 3 |
| 1.3 | Weighted first passage percolation model | 4 |
| 1.4 | Standard FPP limiting shape and weighted metric | 5 |
| 1.5 | Main results | 6 |
| 1.6 | Outline | 10 |
| 2 | General results for weighted FPP | 10 |
| 2.1 | Markov property and applications | 10 |
| 2.2 | Comparison to standard exponential FPP | 12 |
| 2.3 | Weak one-endedness | 13 |
| 3 | Estimating passage times via a deterministic metric | 16 |
| 3.1 | Rate of convergence estimates for standard FPP | 16 |
| 3.2 | Estimates for the weighted metric | 17 |
| 3.3 | Growth estimates for weighted FPP | 18 |

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| | | |
|----------|-------------------------------------------------------------------|-----------|
| 4 | Proof of limit shape and covering results | 23 |
| 4.1 | Proof of Theorem 1.7 | 23 |
| 4.2 | Proof of Theorem 1.9 | 27 |
| 5 | Proof of cone containment result | 29 |
| 5.1 | Cylindrical convex sets and admissible weight functions | 29 |
| 5.2 | Geometric estimates for cylindrical sets | 31 |
| 5.3 | Probabilistic estimates | 34 |
| 5.4 | Proof of Theorem 5.2 | 37 |
| 6 | Open problems | 38 |

1 Introduction

1.1 Overview

Let $d \in \mathbb{N}$ and equip \mathbb{Z}^d with its standard cubic lattice structure. The *Eden model* is a simple statistical physics model introduced in [Ede61], defined as follows. Let e_1 be sampled uniformly from the set of edges of \mathbb{Z}^d incident to 0, and set $\tilde{A}_1 = \{e_1\}$. Inductively, if $n \in \mathbb{N}$, $n \geq 2$, and \tilde{A}_{n-1} has been defined, let e_n be sampled uniformly from the set of edges of \mathbb{Z}^d incident to \tilde{A}_{n-1} and set $\tilde{A}_n := \tilde{A}_{n-1} \cup \{e_n\}$.

The Eden model is equivalent to first passage percolation with iid exponentially distributed edge passage times, which was first introduced in [HW65] (this is a consequence of the “memoryless” property of the exponential distribution). Under this representation, the Eden model has been studied extensively, but many aspects of this model are still poorly understood. For example, it is known that the clusters \tilde{A}_n have a deterministic limiting shape \mathbb{A} in a rather strong sense (see [Ric73, CD81, Kes93] as well as Sections 1.4 and 3.1 below), but little is known about this limit shape besides that it is compact, convex, and satisfies the same symmetries as \mathbb{Z}^d . We refer the reader to the survey articles [Kes86, Kes87, How04, Bla10, GK12, AHD15] and the references therein for more information on first passage percolation.

In this article, we will consider the following natural variant of the Eden model. Let wt be a weight function from the edge set of \mathbb{Z}^d to the positive real numbers. The *weighted Eden model with edge weights* wt is the growing family of edge sets $\{\tilde{A}_n\}_{n \in \mathbb{N}}$ which is defined in the same manner as the Eden clusters above, except that each edge e_n is sampled from the set of edges incident to \tilde{A}_{n-1} with probability proportional to $\text{wt}(e)$ instead of uniformly. Like the standard Eden model, this model can also be expressed in terms of a variant of first passage percolation where the passage time of each edge e is an independent exponential random variable with parameter $\text{wt}(e)$ (in fact, we will mostly focus our attention on this latter model, which seems to be easier to analyze), see Section 1.3.

We will primarily be interested in the following special case of the above model. Fix $\alpha \in \mathbb{R}$. Let $f_0 : \partial\mathbb{D} \rightarrow (0, \infty)$ be a strictly positive Lipschitz function on the boundary of the Euclidean unit ball \mathbb{D} . Let

$$f(z) = |z|^\alpha f_0(z/|z|), \quad \forall z \in \mathbb{R}^d \setminus \{0\} \quad (1.1)$$

so that f is strictly positive, locally Lipschitz, and homogeneous of degree α . We call such a function f an α -*weight function*. A particular example of an α -weight function is the α th power of some norm ν on \mathbb{R}^d , which corresponds to $f_0(z) = \nu(z)^\alpha$ for $z \in \partial\mathbb{D}$. The f -*weighted Eden model* is the weighted Eden model where the weight of each edge e of \mathbb{Z}^d is given by

$$\text{wt}(e) := f(m_e), \quad (1.2)$$

where m_e is the midpoint of e . In the case where $d = 1$ and $f(z) = |z|$, the f -weighted Eden model is a slight variant of the Pólya urn model, so the f -weighted Eden model can be viewed as higher-dimensional generalization of the Pólya urn model. The f -weighted Eden model in the case where $f(z) = |z|^\alpha$ was first introduced as an open problem in [Bub15].

Weighted versions of the eden model have been studied elsewhere in the literature. Diffusion limited aggregation (DLA) on a d -ary tree is equivalent to a weighted variant of the Eden model on the tree with edge weights which are an exponential, rather than polynomial, function of the distance to the root vertex.

This model is studied in [AS88,BPP97]. In the computer science literature, the authors of [FKOV14] propose a weighted version of the Eden model on a general graph, which they call “adaptive diffusion”, as a protocol for spreading a message in a network while obscuring its source.

As we shall see, the asymptotic behavior of f -weighted FPP in general dimension and for general choice of f depends crucially on the homogeneity degree of f . In particular, we will prove the following.

- If $\alpha \in (-\infty, 1)$, the f -weighted FPP clusters (for any choice of weight function f) have a deterministic compact limit shape which is an explicit functional of f and the standard Eden model limit shape \mathbb{A} . We also provide a rate of convergence estimate in the spirit of [Kes93, Ale97].
- If $\alpha > 1$, there exists a norm ν on \mathbb{R}^d depending on α (which we can take to be an explicit functional of \mathbb{A} and α) such that with $f(z) = \nu(z)^\alpha$, the f -weighted FPP clusters are a.s. contained in a certain Euclidean cone with opening angle $< \pi$ at all times.
- For any choice of the Lipschitz function $f_0 : \partial\mathbb{D} \rightarrow (0, \infty)$ in (1.1), there is a constant $c > 0$ (again, depending explicitly on \mathbb{A} and f_0) such that if $\alpha \in [1, 1 + c)$ then a.s. the f -weighted FPP clusters with weight function f eventually hit all but finitely many edges in \mathbb{Z}^d .

See Section 1.5 below for precise statements. We also include several open problems related to the weighted Eden model, see Section 6.

The main idea of our proofs is to compare the local behavior of f -weighted FPP to the local behavior of standard FPP. This allows us to show that passage times in f -weighted FPP are well-approximated by a deterministic metric D , which is defined precisely in Section 1.4 and depends on f and the standard FPP limit shape \mathbb{A} .

Remark 1.1. In the open problem statement [Bub15], it is conjectured that for $f(z) = |z|^\alpha$, the f -weighted FPP clusters a.s. have a deterministic limit shape if $\alpha < 1$ and are a.s. contained in a Euclidean cone of opening angle $< \pi$ at all times if $\alpha > 1$. Our results confirm this conjecture in the case $\alpha < 1$. In the case $\alpha > 1$, our results show that this conjecture is false for α sufficiently close to 1, but is true if we replace $|\cdot|$ with a norm which is allowed to depend on α . It is still an open problem to determine whether it holds for large enough α that the f -weighted FPP clusters with $f(z) = |z|^\alpha$ are a.s. contained in a Euclidean cone of opening angle $< \pi$ for all times.

Remark 1.2. We include several simulations of f -weighted FPP clusters, which are scattered throughout Section 1. All of these simulations are produced using Matlab and are run for 10^6 iterations. Particles are color-coded based on the time at which they are added to the cluster. In order to reduce the file size of the images, we re-sampled a subset of the 10^6 particles in the clusters. This re-sampling does not significantly change the images, except that some of the images include small white dots corresponding to points which are contained in the cluster, but which were removed during the re-sampling.

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1.2 Basic notations

Before stating our main results we record some (mostly standard) notations which we will use throughout this paper.

1.2.1 Intervals and asymptotics

For $a < b \in \mathbb{R}$, we define the discrete intervals $[a, b]_{\mathbb{Z}} := [a, b] \cap \mathbb{Z}$ and $(a, b)_{\mathbb{Z}} := (a, b) \cap \mathbb{Z}$.

If a and b are two quantities, we write $a \preceq b$ (resp. $a \succeq b$) if there is a constant C (independent of the parameters of interest) such that $a \leq Cb$ (resp. $a \geq Cb$). We write $a \asymp b$ if $a \preceq b$ and $a \succeq b$.

If a and b are two quantities which depend on a parameter x , we write $a = o_x(b)$ (resp. $a = O_x(b)$) if $a/b \rightarrow 0$ (resp. a/b remains bounded) as $x \rightarrow 0$ (or as $x \rightarrow \infty$, depending on context). We write $a = o_x^\infty(b)$ if $a = o_x(b^{-s})$ for each $s > 0$.

Unless otherwise stated, all implicit constants in \asymp, \preceq , and \succeq and $O_x(\cdot)$ and $o_x(\cdot)$ errors involved in the proof of a result are required to satisfy the same dependencies as described in the statement of said result.

1.2.2 Graphs

For a graph G , we write $\mathcal{V}(G)$ for the set of vertices of G and $\mathcal{E}(G)$ for the set of edges of G .

For a graph G and a subset E of $\mathcal{E}(G)$ we write ∂E for the set of edges of G not contained in E which are incident to an edge of E . For a subset V of $\mathcal{V}(G)$, we write ∂V for the set of vertices $v \in \mathcal{V}(G)$ which are incident to vertices of G not contained in V .

Let G be a graph and let $n \in \mathbb{N} \cup \{\infty\}$. A *path* of length n in G is a sequence $\eta = \{\eta(i)\}_{i \in [1, n]_{\mathbb{Z}}} \subset \mathcal{E}(G)$ such that the edges $\eta(i)$ can be oriented in such a way that the initial endpoint of $\eta(i)$ coincides with the terminal endpoint of $\eta(i-1)$ for each $i \in [2, n]_{\mathbb{Z}}$. We say that η is *simple* if η does not visit any vertex of G more than once. We write $|\eta| = n$ for the length of η .

1.2.3 Metrics

We will have occasion to consider several different metrics on \mathbb{R}^d and \mathbb{Z}^d . We use the following notation to distinguish these metrics.

Let D be a metric on \mathbb{R}^d . For $r > 0$ and $z \in \mathbb{R}^d$, we write $B_r^D(z)$ for the closed ball of radius r centered at z in the metric D . For a set $A \subset \mathbb{R}^d$, we write $\text{diam}^D(A)$ for the D -diameter of A . If ν is a norm on \mathbb{R}^d , we write $\text{dist}^\nu(z, w) = \nu(z - w)$ for the metric induced by ν . We often abbreviate $B_r^{\text{dist}^\nu}(z) = B^\nu(z)$.

We write $|\cdot|$ for the Euclidean norm on \mathbb{R}^d and $\mathbb{D} := B_1^{|\cdot|}(0)$ for its unit ball.

1.3 Weighted first passage percolation model

In most of this paper we will consider the following weighted variant of first passage percolation instead of the weighted Eden model described above. The two models are shown to be equivalent in Lemma 2.3 below. We first define the model in the greatest possible generality, then describe the special case which is our primary interest.

Definition 1.3. Let G be a connected, countable graph in which all vertices have finite degree. Let v_0 be a marked vertex of G . Let $\text{wt} : \mathcal{E}(G) \rightarrow (0, \infty)$ be a deterministic function which assigns a positive weight to each $e \in \mathcal{E}(G)$. The *first passage percolation (FPP)* clusters on G started from v_0 with weights wt is the random increasing sequence of subgraphs $\{A_t\}_{t \geq 0}$ of G defined as follows.

- For each edge $e \in \mathcal{E}(G)$, let X_e be an exponential random variable with parameter $\text{wt}(e)$. We take the X_e 's to be independent.
- For a path η in G , let $T(\eta) := \sum_{e \in \eta} X_e$. For vertices $u, v \in \mathcal{V}(G)$, we write

$$T(u, v) := \inf \{T(\eta) : \eta \text{ is a path in } G \text{ from } u \text{ to } v\}.$$

- For $t \in [0, \infty)$, let $A_t \subset G$ be the graph defined as follows. The set of vertices $\mathcal{V}(A_t)$ is the set of $v \in \mathcal{V}(G)$ with $T(v_0, v) \leq t$. The set of edges $\mathcal{E}(A_t)$ is the set of $e \in \mathcal{E}(G)$ such that $e \in \eta$ for some path η in G with $\eta(1)$ incident to v_0 and $T(\eta) \leq t$.

For $t \geq 0$ we write \mathcal{F}_t for the σ -algebra generated by $\{A_s\}_{s \in [0, t]}$ and X_e for $e \in \mathcal{E}(A_t)$. We also let

$$\tau_\infty := \inf \{t > 0 : \#A_t = \infty\} = T(v_0, \infty)$$

be the first (possibly infinite) time at which the cluster is infinite.

Note that ordinary first passage percolation with exponential passage times corresponds to the special case when $\text{wt}(e) = 1$ for each $e \in \mathcal{E}(G)$ in Definition 1.3.

We are primarily interested in the following special case of the model of Definition 1.3, which is a continuous-time parametrization of the f -weighted FPP model described in Section 1.1 (see Lemma 2.3 below). Fix $\alpha \in \mathbb{R}$. Let $G = \mathbb{Z}^d$ for $d \in \mathbb{N}$ (with the standard cubic lattice structure) and let $v_0 = 0$. Let $f_0 : \partial\mathbb{D} \rightarrow (0, \infty)$ be a Lipschitz function and let $f(z) = |z|^\alpha f_0(z/|z|)$ be as in (1.1) and $\text{wt}(e) = f(m_e)$ as in (1.2). Let $\{A_t\}_{t \geq 0}$, $T(\cdot)$, and τ_∞ be as in Definition 1.3 with this choice of parameters. We call the above model *f-weighted FPP*. We also introduce the notation

$$\bar{\kappa} := \sup_{z \in \partial\mathbb{D}} f_0(z) \quad \text{and} \quad \underline{\kappa} := \inf_{z \in \partial\mathbb{D}} f_0(z). \quad (1.3)$$

We note that it is easy to see (by considering a path from 0 to ∞ along a coordinate axis) that for our model $\tau_\infty < \infty$ a.s. whenever $\alpha > 1$. It will follow from Theorem 1.7 (resp. the proof of Theorem 1.9) below that a.s. $\tau_\infty = \infty$ whenever $\alpha < 1$ (resp. $\alpha = 1$).

1.4 Standard FPP limiting shape and weighted metric

Our main method for studying the model described in Section 1.3 is to compare it to standard FPP, i.e. the case where $f \equiv 1$, which is equivalent to the unweighted Eden model. In this case, it is shown in [Ric73, CD81] that there exists a compact convex set $\mathbb{A} \subset \mathbb{R}^d$ which is symmetric about 0 such that the random sets $t^{-1}A_t$ converge a.s. as $t \rightarrow \infty$ to \mathbb{A} in the following sense. For $t > 0$, let

$$A_t^F := \{v + z : v \in \mathcal{V}(A_t), z \in [-1/2, 1/2]^d\} \quad (1.4)$$

be the “fattening” of A_t , so that A_t^F contains no isolated points and $A_t^F \cap \mathbb{Z}^d = \mathcal{V}(A_t)$. Then for each $\epsilon > 0$,

$$\lim_{t \rightarrow \infty} \mathbb{P}((1 - \epsilon)\mathbb{A} \subset s^{-1}A_s^F \subset (1 + \epsilon)\mathbb{A}, \quad \forall s \geq t) = 1. \quad (1.5)$$

Not much is known rigorously about the limit shape \mathbb{A} besides that it is compact, convex, and has the same symmetries as \mathbb{Z}^d . It is expected that \mathbb{A} is not the Euclidean unit ball, but even this is not known except in dimension $d \geq 35$ [CEG11]. See, e.g., [FSS85, BH91, ED14] for numerical studies of Eden clusters.

Let μ be the norm whose closed unit ball is \mathbb{A} , i.e.

$$\mu(z) := \inf\{r > 0 : z \in r\mathbb{A}\}, \quad \forall z \in \mathbb{R}^d. \quad (1.6)$$

We will have occasion to compare \mathbb{A} to the Euclidean unit ball. For this purpose we use the following notation.

Definition 1.4. Let

$$\bar{\rho} := \sup_{z \in \mathbb{R}^d \setminus \{0\}} \frac{|z|}{\mu(z)} \quad \text{and} \quad \underline{\rho} := \inf_{z \in \mathbb{R}^d \setminus \{0\}} \frac{|z|}{\mu(z)}. \quad (1.7)$$

Also let

$$\mathbb{X} := \{x \in \partial\mathbb{A} : \rho_x = \bar{\rho}\} \quad (1.8)$$

be the set of points on $\partial\mathbb{A}$ furthest from 0.

In the remainder of this subsection, we will define a metric $D = “f^{-1} \cdot \mu”$ on $\mathbb{R}^d \setminus \{0\}$ which will turn out to be a good approximation for passage times in our weighted FPP model.

Definition 1.5. A *piecewise linear path* in \mathbb{R}^d is a continuous map $\gamma : [0, T] \rightarrow \mathbb{R}^d$ for some $T > 0$ for which there exists a subdivision $0 = t_0 < \dots < t_n = T$ of $[0, T]$ such that $\gamma|_{[t_{k-1}, t_k]}$ is affine for each $k \in [1, n]_{\mathbb{Z}}$. We say that γ is *parametrized by μ -length* if the following is true. For $t \in [0, T]$, let K_t be the largest $k \in [1, n]_{\mathbb{Z}}$ with $t_k \leq t$. Then

$$t = \mu(\gamma(t) - \gamma(t_{K_t})) + \sum_{k=1}^{K_t} \mu(\gamma(t_k) - \gamma(t_{k-1})),$$

i.e. t is the sum of the μ -lengths of the linear segments of γ traced up to time t . In this case we write $\text{len}^\mu(\gamma) = T$.

If $\gamma : [0, T] \rightarrow \mathbb{R}^d$ is a piecewise linear path parametrized by μ -length, we define the D -length of γ by

$$\text{len}^D(\gamma) := \int_0^T f(\gamma(t))^{-1} dt, \quad (1.9)$$

with f the α -weight function from (1.1). If γ is not necessarily parametrized by μ -length, we define the D -length of γ to be the D -length of the path obtained by parametrizing γ by μ -length. We define a metric on \mathbb{R}^d by

$$D(z, w) := \inf_{\gamma} \text{len}^D(\gamma) \quad \forall z, w \in \mathbb{R}^d \quad (1.10)$$

where the infimum is over all piecewise linear paths γ connecting z and w .

As we shall see in Section 3 below, $D(z, w)$ is a good approximation for the passage time $T(z, w)$ in the f -weighted FPP process $\{A_t\}_{t \geq 0}$. The following lemma is immediate from the α -homogeneity of f and the definition (1.10) of D .

Lemma 1.6. *Let $z, w \in \mathbb{R}^d$ and $r > 0$. Then*

$$D(rz, rw) = r^{1-\alpha} D(z, w). \quad (1.11)$$

1.5 Main results

Throughout this section, we assume that we are in the special case of Definition 1.3 described in Section 1.3, so in particular $\alpha \in \mathbb{R}$, f is an α -weight function as in (1.1), and $\{A_t\}_{t \geq 0}$ are the f -weighted FPP clusters.

Let D be the metric from Section 1.4. If $\alpha < 1$, then it is easy to see by integration that $\lim_{w \rightarrow 0} D(w, z)$ is finite for each $z \in \mathbb{R}^d \setminus \{0\}$ and that D extends to a metric on all of \mathbb{R}^d . In particular, the D -balls $B_r^D(0)$ for $r > 0$ are well-defined. Let $\mathbb{B} = \mathbb{B}_f := B_1^D(0)$. We note that Lemma 1.6 implies that

$$B_r^D(0) = r^{\frac{1}{1-\alpha}} \mathbb{B}, \quad \forall r > 0. \quad (1.12)$$

The set \mathbb{B} is the limiting shape of the f -weighted FPP clusters for $\alpha < 1$, in the following sense.

Theorem 1.7. *Let $\alpha \in (-\infty, 1)$ and*

$$\chi \in \left(0, \frac{1}{3(1-\alpha)}\right). \quad (1.13)$$

For $t > 0$, let A_t^F be as in (1.4) (for a general choice of f_0). Then for $t_0 > 0$,

$$\mathbb{P} \left((1 - t^{-\chi}) t^{\frac{1}{1-\alpha}} \mathbb{B} \subset A_t^F \subset (1 + t^{-\chi}) t^{\frac{1}{1-\alpha}} \mathbb{B} \text{ for all } t \geq t_0 \right) = 1 - o_{t_0}^\infty(t_0),$$

where here $o_{t_0}^\infty(t_0)$ denotes a quantity which decays faster than any negative power of t_0 as $t_0 \rightarrow \infty$ (recall Section 1.2.1).

Theorem 1.7 gives in some sense a complete qualitative characterization of the asymptotic behavior of the f -weighted FPP clusters when $\alpha < 1$. However, we expect that the exponent χ in (1.13) is not optimal (in fact, we expect the theorem to be true at least for any $\chi \in \left(0, \frac{2}{5(1-\alpha)}\right)$; c.f. Remark 3.1 below). Moreover, we cannot give a more explicit description of the limit shape \mathbb{B} than the one above. Indeed, we cannot even characterize the functions f for which the set \mathbb{B} is convex. See Figures 1 and 2 for simulations of f -weighted FPP clusters with $\alpha < 1$, some of which appear to have a non-convex limit shape.

In the case $\alpha \geq 1$, matters are more complicated. The qualitative asymptotic behavior of the f -weighted FPP clusters depends crucially on the function f , rather than just the value of α . In the case when $\alpha > 1$, simulations like the ones in Figure 3 suggest that the f -weighted FPP clusters for many choices of f tend to grow in a single direction, rather than being ball-like like in the case when $\alpha < 1$. We recall that $\tau_\infty = \inf\{t \geq 0 : \#\mathcal{E}(A_t) = \infty\}$. Our next theorem tells us that for each $\alpha > 1$, there exists a norm on \mathbb{R}^d (depending on α) such that if f is the α th power of this norm, then A_{τ_∞} is a.s. contained in a cone of opening angle $< \pi$.

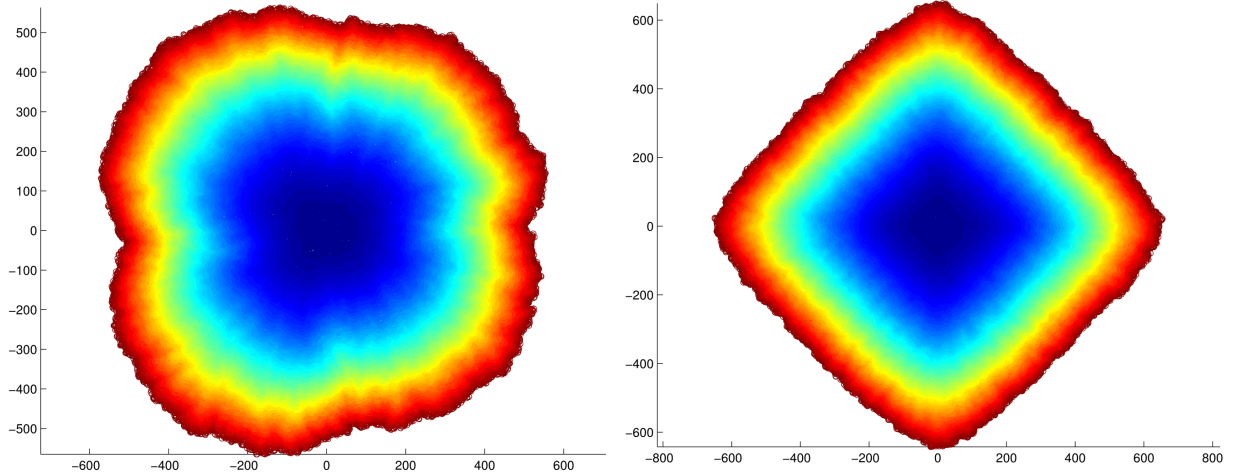


Figure 1: **Left panel:** A simulation of an f -weighted FPP cluster with $f(z) = \|z\|_1^{1/2}$, where here $\|\cdot\|_1$ is the L^1 norm (which restricts to the graph distance on \mathbb{Z}^2). The clusters appear to be converging to a deterministic limit shape (which we know is a.s. the case by Theorem 1.7), but it is not clear from the simulation whether this limit shape is convex. **Right panel:** A simulation of an f -weighted FPP cluster with $f(z) = \|z\|_1^{-2}$, where here $\|\cdot\|_1$ is the L^1 norm. The clusters appear to be converging to a deterministic limit shape which is a slight rounding of the L^1 -unit ball.

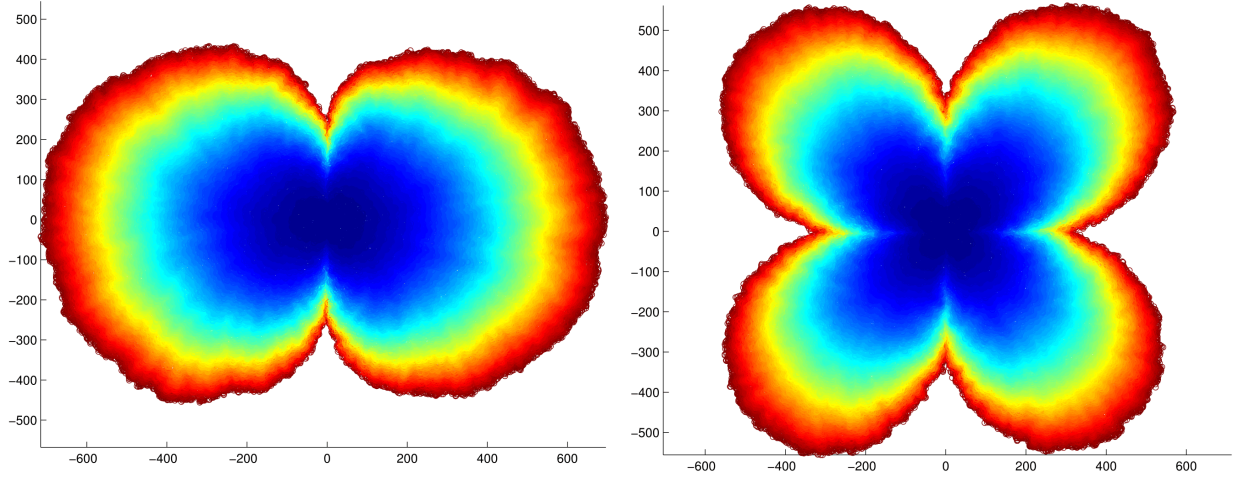


Figure 2: **Left panel:** A simulation of an f -weighted FPP cluster with $f(z) = \nu(z)^{1/2}$, where here ν is the norm whose closed unit ball is the rectangle $[-1, 1] \times [-100, 100]$. **Right panel:** A simulation of an f -weighted FPP cluster with the weight function f given by the third power of the ratio of the L^1 norm to the Euclidean norm (so $\alpha = 0$). We note that in both figures, the limit shape appears to be non-convex.

Theorem 1.8. *For each $\alpha > 1$ and each $\mathfrak{x} \in \mathbb{X}$ (Definition 1.4), there exists a norm $\nu = \nu(\alpha, \mathfrak{x})$ on \mathbb{R}^d and a $\theta \in (0, \pi)$ such that the following is true. Let*

$$\mathcal{C} = \left\{ z \in \mathbb{R}^d : \left\langle \frac{z}{|z|}, \frac{\mathfrak{x}}{|\mathfrak{x}|} \right\rangle > \cos \theta \right\}$$

be the Euclidean cone based at 0 with opening angle θ centered at the ray from 0 through \mathfrak{x} . Also let $f(z) := \nu(z)^\alpha$ and let $\{A_t\}_{t \geq 0}$ the f -weighted FPP process. Then a.s. either

$$\#(\mathcal{V}(A_{\tau_\infty}) \setminus \mathcal{C}) < \infty \quad \text{or} \quad \#(\mathcal{V}(A_{\tau_\infty}) \setminus (-\mathcal{C})) < \infty. \quad (1.14)$$

We will actually prove a more quantitative version of Theorem 1.8 (see Theorem 5.2 below). This result says that the statement of Theorem 1.8 holds for all α -weight functions f satisfying certain conditions, which are satisfied for the α -th powers of a certain class of norms on \mathbb{R}^d . The unit ball of a typical norm in this class is a “cylinder” of the form $\{sz + t\mathbf{x} : z \in Q, t \in [-1, 1]\}$ where Q is a compact convex subset of the hyperplane through the origin perpendicular to \mathbf{x} and s is a large fixed parameter which tends to ∞ as $\alpha \rightarrow 1^+$. See Figure 4 for an illustration.

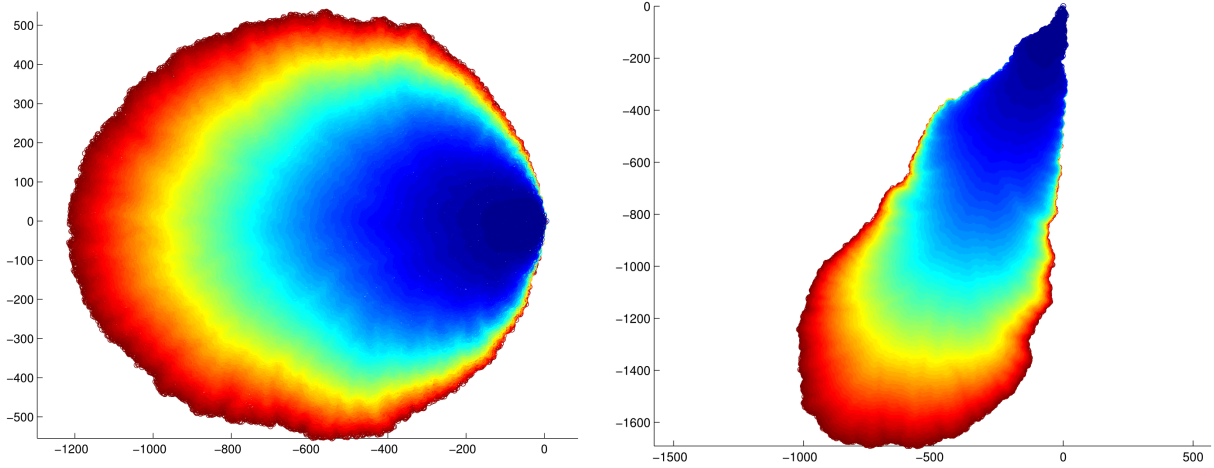


Figure 3: **Left panel:** A simulation of an f -weighted FPP cluster with $f(z) = \nu(z)^{1.1}$, where here ν is the norm whose closed unit ball is the rectangle $[-1, 1] \times [-100, 100]$. This norm ν is similar to the norm appearing in Theorem 1.8, although in Theorem 1.8 the rectangle may be rotated by some (non-explicit) angle which depends on the standard FPP limit shape \mathbb{A} . **Right panel:** A simulation of an f -weighted FPP cluster with $f(z) = |z|^5$. The figure suggests that the clusters will be contained in a Euclidean cone with opening angle $< \pi$ for all times $t \leq \tau_\infty$, but we do not prove that this is the case for this particular choice of f .

It is an open problem to give for each $\alpha > 1$ a reasonably (though perhaps not fully) general characterization of the choices of f for which the conclusion of Theorem 1.8 holds. We expect that a rigorous proof of such a characterization may require additional knowledge about the standard FPP limit shape \mathbb{A} .

Theorem 1.8 focuses on the behavior of the FPP clusters up to time τ_∞ , which is a.s. finite for $\alpha > 1$. It is natural to ask about the behavior of the clusters A_t for $t > \tau_\infty$. Straightforward tail estimates for sums of exponential random variables (see, e.g. [Jan14, Theorem 5.1, item (i)]) show that if $\alpha > 1$, then it is a.s. the case that for each $\epsilon > 0$, the set $\mathcal{V}(A_{\tau_\infty + \epsilon})$ contains all but finitely many vertices of \mathbb{Z}^d . Hence there is no interesting macroscopic behavior after time τ_∞ .

One may wonder to what extent the norm ν and the cone \mathcal{C} in Theorem 1.8 can taken to be uniform in α . It turns out that the condition on ν needed for (1.14) to hold a.s. differs from the condition needed for this result to hold with positive probability. In particular, our more quantitative statement Theorem 5.2 implies the following.

- For any $\alpha_2 > \alpha_1 > 1$, we can choose ν and \mathcal{C} such that whenever $\alpha \in [\alpha_1, \alpha_2]$ and $f(z) = \nu(z)^\alpha$, the condition (1.14) holds a.s.
- For any $\alpha_1 > 1$, we can choose ν and \mathcal{C} such that whenever $\alpha \geq \alpha_1$, we have that (1.14) holds with positive probability.

We note that Theorem 1.9 below tells us that ν cannot be chosen uniformly for all $\alpha > 1$.

Our next theorem tells us that there is no choice of the function f_0 of (1.1) for which the conclusion of Theorem 1.8 holds for every choice of $\alpha > 1$. In fact, we will show that if $\alpha \geq 1$ is sufficiently close to 1 (depending on f_0), then $\mathcal{V}(A_{\tau_\infty})$ a.s. contains all but finitely many vertices of \mathbb{Z}^d . To quantify how close to 1 we need α to be, we introduce some notation. For $\delta > 0$ and $z, w \in \partial\mathbb{D}$, let $\Gamma_\delta(z, w)$ be the set of

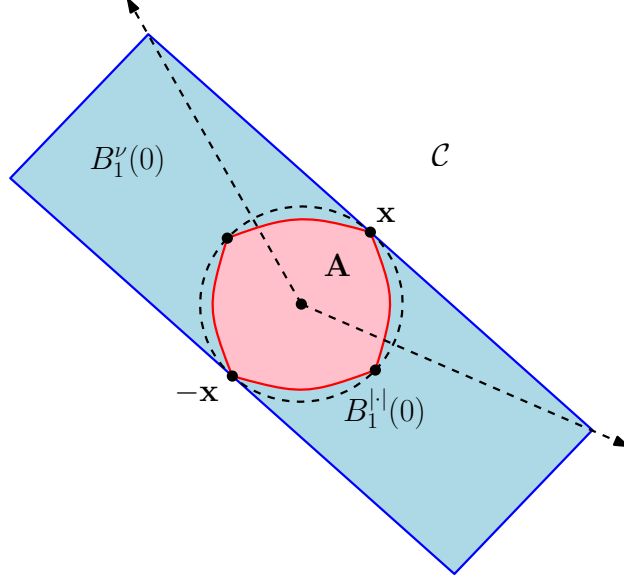


Figure 4: An illustration of a the unit ball $B_1^\nu(0)$ (light blue) of a typical norm ν satisfying the conclusion of Theorem 1.8 when α slightly bigger than 1 and $d = 2$. Also shown is the Eden model limit shape A (pink) and the smallest Euclidean ball which contains it, namely $B_{\bar{\rho}}^{|\cdot|}(0)$ (dashed boundary). The boundary of the cone \mathcal{C} is shown as a pair of dashed lines. As α approaches 1, the opening angle of this cone approaches π . However, we do not prove that the opening angle of \mathcal{C} approaches 0 as $\alpha \rightarrow \infty$.

piecewise linear paths (Definition 1.5) connecting z and w which can be decomposed into linear segments whose endpoints are all contained in $\partial\mathbb{D}$ and which each have Euclidean length at most δ . Let

$$\lambda := \limsup_{\delta \rightarrow 0} \sup_{z, w \in \partial\mathbb{D}} \inf_{\gamma \in \Gamma_\delta(z, w)} \text{len}^D(\gamma) \quad (1.15)$$

be half the D -circumference of $\partial\mathbb{D}$. Since $f \equiv f_0$ on $\partial\mathbb{D}$, it is easy to see that λ depends only on f_0 , not on α , and that $0 < \lambda < \infty$ for any choice of f_0 . Furthermore, if we take $f(z) = \nu(z)^\alpha$ for some norm ν on \mathbb{R}^d , then λ depends on α but is uniformly positive for α in any bounded subset of \mathbb{R} .

Theorem 1.9. *Let $f_0 : \partial\mathbb{D} \rightarrow (0, \infty)$ be the Lipschitz function in (1.1). Let $\bar{\rho}$ be as in (1.7), $\bar{\kappa}$ as in (1.3), and λ as in (1.15). Suppose*

$$1 \leq \alpha < 1 + (\bar{\rho}\bar{\kappa}\lambda)^{-1}.$$

For $r > 0$, let

$$\sigma_r := \inf \left\{ t \geq 0 : A_t \not\subset B_r^{|\cdot|}(0) \right\}. \quad (1.16)$$

There is a constant $R > 1$, depending only on μ and f , such that

$$\mathbb{P} \left(\mathbb{Z}^d \cap \left(B_n^{|\cdot|}(0) \setminus B_{n-1}^{|\cdot|}(0) \right) \subset \mathcal{V}(A_{\sigma_{Rn}}) \right) = 1 - o_n^\infty(n). \quad (1.17)$$

In particular, a.s. $\mathbb{Z}^d \setminus \mathcal{V}(A_{\tau_\infty})$ is a finite set.

Remark 1.10. In the case when $\alpha = 1$, it will be clear from the proof of Theorem 1.9 that a.s. $\tau_\infty = \infty$, so $\mathcal{V}(A_{\tau_\infty}) = \mathbb{Z}^d$.

See Figure 5 for simulations of f -weighted FPP clusters in the setting of Theorem 1.9.

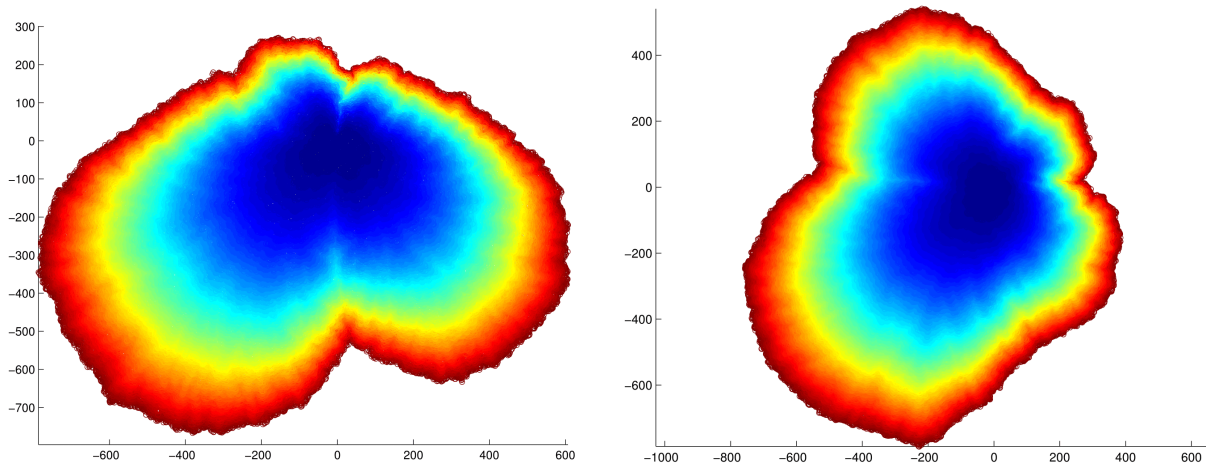


Figure 5: **Left panel:** A simulation of an f -weighted FPP cluster with $f(z) = \|z\|_1^{1.2}$, where here $\|\cdot\|_1$ is the L^1 norm. The figure illustrates the conclusion of Theorem 1.9, namely that the clusters will a.s. cover all but finitely many points of \mathbb{Z}^2 before reaching ∞ . However, these clusters need not grow in a symmetric manner **Right panel:** A simulation of an f -weighted FPP cluster with $f(z) = \|z\|_1$ (so $\alpha = 1$). The clusters do not appear to be converging toward a deterministic limit shape, but it is conceivable that they converge toward a random limit shape or that they converge toward a deterministic limit shape at a very slow rate.

1.6 Outline

The remainder of this paper is structured as follows. In Section 2, we prove some basic properties of the weighted FPP model of Definition 1.3 at a greater level of generality than what we will consider in the remainder of the paper. In Section 3, we prove several lemmas which allow us to approximate f -weighted FPP passage times via the deterministic metric D of (1.10). In Section 4, we use these estimates to prove Theorems 1.7 and Theorem 1.9. In Section 5, we prove Theorem 1.8. In Section 6, we list some open problems related to the model studied in this paper.

2 General results for weighted FPP

Throughout this section we assume we are in the setting of Definition 1.3 for a general choice of graph G , starting vertex v_0 , and weights wt . We recall in particular the FPP clusters $\{A_t\}_{t \geq 0}$ and the FPP filtration $\{\mathcal{F}_t\}_{t \geq 0}$.

In this section we will point out some basic properties of the model of Definition 1.3. In later sections we will only need the case where $G = \mathbb{Z}^d$, $v_0 = 0$, and wt is as in (1.2), but it is no more difficult to treat the general case. In Section 2.1, we state the strong Markov property of our model (which follows from the fact that the passage times have an exponential distribution) and deduce some basic consequences. In Section 2.2, we will prove a lemma which allows us to compare weighted FPP to standard FPP (equivalently, the unweighted Eden model). In Section 2.3, we will prove a weak form of one-endedness for weighted FPP clusters in the case where the graph G is infinite and the passage time to ∞ , τ_∞ , is a.s. finite.

2.1 Markov property and applications

The following lemma gives a Markov property for weighted FPP clusters, and is the reason why we consider exponential passage times.

Lemma 2.1 (Strong Markov property). *Let τ be a stopping time for the FPP filtration $\{\mathcal{F}_t\}_{t \geq 0}$. The conditional law of the passage times of the explored edges, $\{X_e : e \in \mathcal{E}(G \setminus A_\tau)\}$ given \mathcal{F}_τ is described as follows.*

- For $e \in \mathcal{E}(G) \setminus (\mathcal{E}(A_\tau) \cup \partial\mathcal{E}(A_\tau))$, the conditional law of X_e is the same as its marginal law.
- For $e \in \partial\mathcal{E}(A_\tau)$, the conditional law of e is that of an exponential random variable of parameter $\text{wt}(e)$ plus $\tau - \tilde{T}^\tau(e)$, where $\tilde{T}^\tau(e)$ is the minimum of $T(\eta)$ over all paths η in A_τ joining v_0 to an endpoint of e .
- The random variables $\{X_e : e \in \mathcal{E}(G) \setminus \mathcal{E}(A_\tau)\}$ are conditionally independent given \mathcal{F}_τ .

Proof. The case where τ is deterministic follows from the memoryless property of exponential random variables. From this, we immediately obtain the case where τ takes on only countably many possible values. The case of a general stopping time τ is proven by approximating τ by a sequence of stopping times which take on only countably many possible values. \square

Lemma 2.1 motivates the following definition.

Definition 2.2. For $t \in \mathbb{R}$ and an edge $e \in \mathcal{E}(G) \setminus \mathcal{E}(A_t)$, let

$$\hat{X}_e^t := \begin{cases} X_e & e \notin \partial\mathcal{E}(A_t) \\ X_e - t + \tilde{T}^t(e) & e \in \partial\mathcal{E}(A_t) \end{cases} \quad (2.1)$$

where $\tilde{T}^t(e)$ is as in Lemma 2.1. For a path η in $G \setminus A_\tau$, let

$$\hat{T}^t(\eta) := \sum_{i=1}^{|\eta|} \hat{X}_{\eta(i)}^t. \quad (2.2)$$

By Lemma 2.1, if τ is a stopping time for the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, then the conditional law given \mathcal{F}_τ of $\{\hat{X}_e^\tau\}_{e \in \mathcal{E}(G)}$ is that of a collection of independent exponential random variables where each \hat{X}_e^τ has parameter $\text{wt}(e)$. Furthermore, if η is a path in $\mathcal{E}(G) \setminus A_t$ with only one edge lying in $\partial\mathcal{E}(A_t)$, then

$$\hat{T}^t(\eta) = T(\eta) - t + \tilde{T}^t(\eta(1)). \quad (2.3)$$

Lemma 2.1 easily implies the following, which gives the equivalence of the model of Definition 1.3 and the weighted Eden model described in Section 1.1.

Lemma 2.3. Assume we are in the setting of Definition 1.3 with $\text{wt}(e) > 0$ for each $e \in \mathcal{E}(G)$. Let $t_0 = 0$ and for $n \in \mathbb{N}$, let t_n be the smallest $t \geq 0$ for which $\#A_t \geq n + 1$. Let $\tilde{A}_n := \mathcal{E}(A_{t_n})$. Then the law of the sequence of random sets $\{\tilde{A}_n\}_{n \in \mathbb{N}}$ is described as follows. Let $\tilde{A}_0 := \{\emptyset\}$. Let e_1 be chosen uniformly from the set of edges of G incident to v_0 and let $\tilde{A}_1 := \{e_1\}$. Inductively, if $n \in \mathbb{N}$ and \tilde{A}_{n-1} has been defined, let e_n be sampled from the uniform measure on the set of edges adjacent to \tilde{A}_{n-1} weighted by wt . Let $\tilde{A}_n := \tilde{A}_{n-1} \cup \{e_n\}$.

We next record another application of the random variables of Definition 2.2, namely a monotonicity statement for realizations of the cluster A_τ when τ is a stopping time for $\{\mathcal{F}_t\}_{t \geq 0}$.

Lemma 2.4. Let τ be a stopping time for $\{\mathcal{F}_t\}_{t \geq 0}$. Let V be a subset of $\mathcal{V}(G)$ chosen in a manner which is measurable with respect to A_τ . Let σ be the smallest $t \geq 0$ for which $\mathcal{V}(A_t) \cap V \neq \emptyset$.

1. $\sigma - \tau$ is conditionally independent from \mathcal{F}_τ given A_τ .
2. Let \mathfrak{A} and \mathfrak{A}' be two possible realizations of A_τ such that $\mathfrak{A} \subset \mathfrak{A}'$ and the realizations of V corresponding to \mathfrak{A} and \mathfrak{A}' are the same. Then the conditional law of $\sigma - \tau$ given $\{A_\tau = \mathfrak{A}\}$ stochastically dominates the conditional law of $\sigma - \tau$ given $\{A_\tau = \mathfrak{A}'\}$.

Proof. First we prove assertion 1. Let H be the set of simple paths η for which the following is true.

1. η connects v_0 to a vertex in V .
2. η contains exactly one edge in $\partial\mathcal{E}(A_\tau)$.

3. Let i_η be the time i for which $\eta(i_\eta) \in \partial\mathcal{E}(A_\tau)$. There is no path $\tilde{\eta}$ in A_τ whose last edge shares an endpoint with $\eta(i_\eta)$ and which satisfies $T(\tilde{\eta}) < T(\eta|_{[1, i_\eta-1]})$.

For $\eta \in H$, we write

$$\hat{\eta} := \eta|_{[i_\eta, |\eta|]_{\mathbb{Z}}}.$$

Then a.s. $\sigma = \min_{\eta \in H} T(\eta)$.

Define the random variables \hat{X}_e^τ for $e \in \mathcal{E}(G) \setminus \mathcal{E}(A_\tau)$ and the passage times $\tilde{T}^\tau(\cdot)$ and $\hat{T}^\tau(\cdot)$ as in Definition 2.2. Note that condition 3 in the definition of H implies that $\tilde{T}^\tau(\eta(i_\eta)) = T(\eta \setminus \hat{\eta})$. Hence for $\eta \in H$,

$$T(\eta) = T(\hat{\eta}) + T(\eta \setminus \hat{\eta}) = T(\hat{\eta}) + \tilde{T}^\tau(\eta(i_\eta)).$$

By (2.3), we obtain $T(\eta) - \tau = \hat{T}^\tau(\hat{\eta})$. Therefore,

$$\sigma - \tau = \min_{\eta \in H} \hat{T}^\tau(\hat{\eta}) \quad (2.4)$$

is a deterministic functional of the set V and the random variables \hat{X}_e^τ for $e \in \mathcal{E}(G) \setminus \mathcal{E}(A_\tau)$. By Lemma 2.1, the conditional law of this latter collection of random variables given \mathcal{F}_τ depends only on A_τ , so this collection of random variables is conditionally independent from \mathcal{F}_τ given A_τ . We thus obtain assertion 1.

Now suppose we are in the setting of assertion 2. Let \hat{H} be the set of simple paths $\hat{\eta}$ whose first edge belongs to $\partial\mathcal{E}(\mathfrak{A})$, none of whose other edges belong to $\partial\mathcal{E}(\mathfrak{A})$, and whose last edge is incident to a vertex in V . In the notation introduced at the beginning of the proof, \hat{H} is the set of paths $\hat{\eta}$ for $\eta \in H$ on the event $\{A_\tau = \mathfrak{A}\}$. Define \hat{H}' similarly but with \mathfrak{A}' in place of \mathfrak{A} . For $\hat{\eta} \in \hat{H}$, let i' be the largest $i \in [1, |\hat{\eta}|]_{\mathbb{Z}}$ with $\hat{\eta}(i) \in \partial\mathfrak{A}'$ and let $\hat{\eta}' := \hat{\eta}|_{[i', |\hat{\eta}|]_{\mathbb{Z}}}$. Then $\hat{\eta}' \in \hat{H}'$.

Let $\{\hat{X}_e : e \in \mathcal{E}(G)\}$ be a collection of independent exponential random variables, each with parameter $\text{wt}(e)$. For a path η in G , let $\hat{T}(\eta) := \sum_{e \in \eta} \hat{X}_e$. By (2.4), the conditional law of $\sigma - \tau$ given $\{A_\tau = \mathfrak{A}\}$ (resp. $\{A_\tau = \mathfrak{A}'\}$) is the same as the law of

$$\min_{\hat{\eta} \in \hat{H}} \hat{T}(\hat{\eta}) \quad \left(\text{resp.} \quad \min_{\hat{\eta}' \in \hat{H}'} \hat{T}(\hat{\eta}') \right).$$

Since $\hat{\eta} \mapsto \hat{\eta}'$ is a surjective map from \hat{H} to \hat{H}' , we obtain the desired stochastic domination. \square

2.2 Comparison to standard exponential FPP

In this subsection, we will record some observations which allow us to compare the model of Section 2.2 to standard FPP on G (i.e. with all of the edge weights $\text{wt}(e)$ equal to 1). For this purpose we first define a collection of iid exponential random variables which are related to the weighted FPP passage times X_e .

Definition 2.5. For $t \geq 0$ and an edge $e \in \mathcal{E}(G \setminus A_t)$, let $\bar{X}_e^t := \text{wt}(e) \hat{X}_e^t$, with \hat{X}_e^t as in Definition 2.2. Also let $\{\bar{X}_e^t\}_{e \in \mathcal{E}(A_t)}$ be a collection of random variables whose conditional law given \mathcal{F}_t is that of a family of iid exponential random variables with parameter 1, independent from the random variables \hat{X}_e^t for $e \in \mathcal{E}(G \setminus A_t)$. For a path η in G , let

$$\bar{T}^t(\eta) := \sum_{i=1}^{|\eta|} \bar{X}_{\eta(i)}^t. \quad (2.5)$$

For $v \in \mathcal{V}(G)$, also let $\{\bar{A}_{v,s}^t\}_{s \geq 0}$ be the FPP clusters started from v corresponding to the collection of random variables $\{\bar{X}_e^t\}_{e \in \mathcal{E}(G)}$, i.e. $e \in \mathcal{E}(\bar{A}_{v,s}^t)$ if and only if there is a path η in $\mathcal{E}(G)$ joining v to e with $\bar{T}^t(\eta) \leq s$ and $\mathcal{V}(\bar{A}_{v,s}^t)$ is the set of endpoints of edges in $\mathcal{E}(\bar{A}_{v,s}^t)$.

We also define an *FPP geodesic* from $v \in \mathcal{V}(G)$ to $e \in \mathcal{E}(G)$ to be a path η in G such that $\eta(1)$ is incident to v , $\eta(|\eta|) = e$, and $T(\eta)$ is minimal among all such paths. If we do not specify the point v , we assume $v = v_0$ is the root vertex of G . It is easy to see that there a.s. exists at most one FPP geodesic from v to e .

Lemma 2.6. *Let τ be a stopping time for the FPP filtration $\{\mathcal{F}_t\}_{t \geq 0}$ and define the random variables \overline{X}_e^τ for $e \in \mathcal{E}(G)$ and the clusters $\overline{A}_{v,t}^\tau$ for $t \geq 0$ as in Definition 2.5. Then the conditional law of $\{\overline{X}_e^\tau\}_{e \in \mathcal{E}(G)}$ given \mathcal{F}_τ is that of a collection of iid exponential random variables, each of which has parameter 1. If $s > 0$ and $e \in \mathcal{E}(G) \setminus \mathcal{E}(A_\tau)$ then the following holds.*

1. *Suppose $e \in \mathcal{E}(A_{\tau+s})$. Let $\eta_e : [1, |\eta_e|]_{\mathbb{Z}} \rightarrow \mathcal{E}(G)$ be the FPP geodesic from v_0 to e . Let v be the last vertex in $\partial\mathcal{V}(A_\tau)$ crossed by η_e and let $\overline{m} := \max_{e' \in \eta_e \setminus \mathcal{E}(A_\tau)} \text{wt}(e')$. Then $e \in \mathcal{E}(\overline{A}_{v,\overline{m}s}^\tau)$.*
2. *Suppose there exists a simple path η in G started from v_0 such that $e \in \eta$, $\eta \cap \partial\mathcal{E}(A_\tau) \neq \emptyset$, and with $\overline{T}^\tau(\eta)$ as in (2.5),*

$$\left(\min_{e' \in \eta} \text{wt}(e') \right)^{-1} \overline{T}^\tau(\eta) \leq s.$$

Then $e \in \mathcal{E}(A_{\tau+s})$.

Proof. From the strong Markov property (Lemma 2.1), Definition 2.5, and the scaling property of exponential random variables, it is clear that the conditional law of $\{\overline{X}_e^\tau\}_{e \in \mathcal{E}(G)}$ given \mathcal{F}_τ is as claimed.

Now suppose the hypotheses of assertion 1 are satisfied. Let i_e be the (a.s. unique) integer $i \in [1, |\eta_e|]_{\mathbb{Z}}$ for which $\eta_e(i) \in \partial\mathcal{E}(A_\tau)$. Then $\eta_e \setminus \mathcal{E}(A_\tau) = \eta_e([i_e, |\eta_e|]_{\mathbb{Z}})$ is a simple path in $\mathcal{E}(G)$ and only its first edge belongs to $\partial\mathcal{E}(A_\tau)$. Since $e \in \mathcal{E}(A_{\tau+s})$, (2.3) implies that (with $\hat{T}^\tau(\cdot)$ as in Lemma 2.1)

$$\tau + s \geq T(\eta_e) = T(\eta_e \setminus \mathcal{E}(A_\tau)) + T(\eta_e \cap \mathcal{E}(A_\tau)) = T(\eta_e \setminus \mathcal{E}(A_\tau)) + \tilde{T}^\tau(\eta_e(i_e)) = \hat{T}^\tau(\eta_e \setminus \mathcal{E}(A_\tau)) + \tau.$$

Hence

$$s \geq \hat{T}^\tau(\eta_e \setminus \mathcal{E}(A_\tau)) \geq \overline{m}^{-1} \overline{T}^\tau(\eta_e \setminus \mathcal{E}(A_\tau)).$$

Therefore $e \in \mathcal{E}(\overline{A}_{v,\overline{m}s}^\tau)$.

Finally, we consider the setting of assertion 2. Let i_1 be the largest $i \in [1, |\eta|]_{\mathbb{Z}}$ such that $\eta(i) \in \partial\mathcal{E}(A_\tau)$ and let $\eta_1 := \eta|_{[i_1, |\eta|]_{\mathbb{Z}}}$. Then η_1 is a simple path whose first edge belongs to $\partial\mathcal{E}(A_\tau)$, none of whose other edges belong to $\partial\mathcal{E}(A_\tau)$, and one of whose edges is e . We have

$$\hat{T}^\tau(\eta_1) \leq \hat{T}^\tau(\eta) \leq \left(\min_{e' \in \eta} \text{wt}(e') \right)^{-1} \overline{T}^\tau(\eta) \leq s.$$

By (2.3), $T(\eta_1) \leq s + \tau - \tilde{T}^\tau(\eta(i_1))$. By definition of \tilde{T}^τ (see Lemma 2.1), there is a path η_0 in A_τ joining 0 to an endpoint of $\eta_1(i_1)$ which satisfies $T(\eta_0) = \tilde{T}^\tau(\eta(i_1))$. If we let η_* be the concatenation of η_0 and η_1 , then $T(\eta_*) \leq \tau + s$, so $e \in \mathcal{E}(A_{\tau+s})$. \square

2.3 Weak one-endedness

The purpose of this subsection is to prove the following weak form of “one-endedness” for the clusters $\{\mathcal{F}_t\}_{t \geq 0}$, which will be used in the proof of Theorem 1.8 to rule out the possibility that all but finitely points of A_{τ_∞} are contained in the disjoint union of two cones of opening angle $< \pi$, rather than a single such cone.

Proposition 2.7. *Suppose we are in the setting of Definition 1.3 with G infinite and the weights wt are chosen in such a way that $\tau_\infty < \infty$ a.s. Let τ be a stopping time for $\{\mathcal{F}_t\}_{t \geq 0}$ with $\tau < \tau_\infty$ a.s. Let Γ_1 and Γ_2 be infinite subgraphs of G which lie at graph distance at least 3 from one another, each of which shares a vertex with A_τ , chosen in some \mathcal{F}_τ -measurable manner (i.e. Γ_1 and Γ_2 are \mathcal{F}_τ -measurable random variables). Then*

$$\mathbb{P}(\#\mathcal{E}(A_{\tau_\infty} \cap \Gamma_1) = \#\mathcal{E}(A_{\tau_\infty} \cap \Gamma_2) = \infty, \#(\mathcal{E}(A_{\tau_\infty}) \setminus \mathcal{E}(\Gamma_1 \cap \Gamma_2)) < \infty | \mathcal{F}_\tau) = 0. \quad (2.6)$$

We note that Proposition 2.7 is *not* sufficient to conclude that A_{τ_∞} is a.s. one ended (i.e., that for large enough n the set of edges of A_{τ_∞} which are not contained in the graph distance ball of radius n centered at the starting vertex is connected). For this to be the case we would need (2.6) to hold simultaneously a.s. for every choice of Γ_1 and Γ_2 . However, Proposition 2.7 is sufficient for the proof of Theorem 1.8.

Proposition 2.7 is proven via a purely probabilistic argument. See Figure 6 for an outline of the proof. We first need the following elementary lemma.

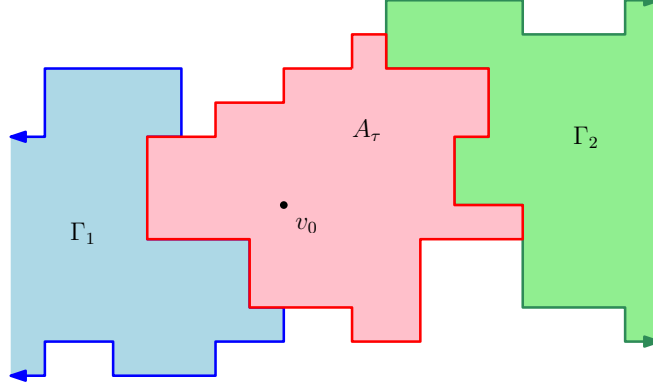


Figure 6: An illustration of a the setup for Proposition 2.7 with $G = \mathbb{Z}^d$. To prove the proposition, we first prove a version of (2.6) where we require that $\#(\mathcal{E}(A_{\tau_\infty}) \setminus \mathcal{E}(A_\tau \cap (\Gamma_1 \cup \Gamma_2)))$ is empty, rather than finite. Let σ_1 (resp. σ_2) be the first time that A_t covers infinitely many edges of Γ_1 (resp. Γ_2) and let E_1 (resp. E_2) be the event that there is no path in A_{σ_1} (resp. A_{σ_2}) which crosses the boundary of Γ_1 (resp. Γ_2). Lemma 2.1 implies that $\sigma_1 \mathbb{1}_{E_1}$ and $\sigma_2 \mathbb{1}_{E_2}$ are conditionally independent given \mathcal{F}_τ . Hence if $\sigma_1 \mathbb{1}_{E_1} = \sigma_2 \mathbb{1}_{E_2} = \tau_\infty$ with positive conditional probability given \mathcal{F}_τ , then the conditional law of τ_∞ given \mathcal{F}_τ must have an atom with positive probability. This contradicts Lemma 2.9, and we obtain the desired weaker version of (2.6). The full version is obtained by applying the weaker version to countably many stopping times between τ and τ_∞ which increase to τ_∞ .

Lemma 2.8. *Let X_1, X_2 , and Y be random variables taking values in a common state space \mathcal{X} . Suppose that X_1 and X_2 are independent and that*

$$\mathbb{P}(X_1 = X_2 = Y) > 0. \quad (2.7)$$

Then there is a deterministic $x \in \mathcal{X}$ such that

$$\mathbb{P}(X_1 = X_2 = Y = x) > 0. \quad (2.8)$$

Proof. For $i \in \{1, 2\}$, let \mathcal{A}_i be the set of atoms of the law of X_i , i.e. the set of $x \in \mathcal{X}$ such that $\mathbb{P}(X_i = x) > 0$. We first claim that it is a.s. the case that on the event $\{X_1 = X_2\}$, the common value of X_1 and X_2 belongs to $\mathcal{A}_1 \cap \mathcal{A}_2$. To see this, we observe that by independence,

$$\mathbb{P}(X_1 = X_2 | X_2) \mathbb{1}_{(X_2 \in \mathcal{X} \setminus \mathcal{A}_1)} = 0,$$

so

$$\mathbb{P}(X_1 = X_2 \in \mathcal{X} \setminus \mathcal{A}_1) = 0.$$

By symmetry, also $\mathbb{P}(X_1 = X_2 \in \mathcal{X} \setminus \mathcal{A}_2) = 0$. Hence (2.7) implies that

$$\mathbb{P}(X_1 = X_2 = Y \in \mathcal{A}_1 \cap \mathcal{A}_2) > 0.$$

The set $\mathcal{A}_1 \cap \mathcal{A}_2$ is countable, so there must exist $x \in \mathcal{A}_1 \cap \mathcal{A}_2$ for which (2.8) holds. \square

Lemma 2.9. *Suppose we are in the setting of Definition 1.3 with G infinite and the weights wt are such that a.s. $\tau_\infty < \infty$. a.s. Let τ be a stopping time for $\{\mathcal{F}_t\}_{t \geq 0}$ with $\tau < \tau_\infty$ a.s. Almost surely, the conditional law given \mathcal{F}_τ of the random variable $\tau_\infty - \tau$ is non-atomic, i.e.*

$$\mathbb{P}(\tau_\infty - \tau = t | \mathcal{F}_\tau) = 0, \quad \forall t \geq 0.$$

Roughly speaking, the idea of the proof is to write $\tau_\infty - \tau = \tau_\infty - \tau' + \tau' - \tau$, where τ' is the smallest time $t > \tau'$ at which another edge is added to the cluster. The conditional law of τ' given \mathcal{F}_τ is non-atomic since it is the minimum of finitely many exponential random variables, and $\tau_\infty - \tau'$ is “almost” conditionally

independent from $\tau' - \tau$ given \mathcal{F}_τ due to the strong Markov property (Lemma 2.1). However, $\tau' - \tau$ and $\tau_\infty - \tau'$ are not quite conditionally independent since the law of $\tau_\infty - \tau'$ depends on the particular realization of $A_{\tau'}$, which in turn might depend on $\tau' - \tau$, so slightly more work is needed.

Proof of Lemma 2.9. Suppose by way of contradiction that the statement of the lemma is false, i.e. the conditional law of $\tau_\infty - \tau$ given \mathcal{F}_τ has an atom with positive probability. Let S be an \mathcal{F}_τ -measurable random variable chosen in such a way that

$$\mathbb{P}(\tau_\infty - \tau = S) > 0. \quad (2.9)$$

For example, S could be the location of the largest atom of the conditional law of $\tau_\infty - \tau$ given \mathcal{F}_τ if it exists (with ties broken in some arbitrary \mathcal{F}_τ -measurable manner) or $S \equiv 0$ if no such atom exists.

Let τ' be the smallest $t > \tau$ for which $A_t \neq A_\tau$. If $\tau_\infty - \tau = S$, then $\tau_\infty - \tau' = S - (\tau' - \tau)$. The random variable

$$S' := S - (\tau' - \tau) \quad (2.10)$$

is $\mathcal{F}_{\tau'}$ -measurable and by (2.9), $\mathbb{P}(\tau_\infty - \tau' = S') > 0$. There are only countably many possible realizations of $A_{\tau'}$, so we can find a positive-probability realization \mathfrak{A}' of $A_{\tau'}$ such that

$$\mathbb{P}(\tau_\infty - \tau' = S' \mid A_{\tau'} = \mathfrak{A}') > 0.$$

Since S' is $\mathcal{F}_{\tau'}$ -measurable, Lemma 2.4 implies that the random variables $\tau_\infty - \tau'$ and S' are conditionally independent given $\{A_{\tau'} = \mathfrak{A}'\}$. By Lemma 2.8 (applied with $X_1 = Y = \tau_\infty - \tau'$ and $X_2 = S'$) there exists a deterministic $t' > 0$ such that

$$\mathbb{P}(\tau_\infty - \tau' = S' = t' \mid A_{\tau'} = \mathfrak{A}') > 0.$$

In particular $\mathbb{P}(S' = t') > 0$, so (recall (2.10))

$$\mathbb{P}(\mathbb{P}(\tau' - \tau = S - t' \mid \mathcal{F}_\tau) > 0) > 0.$$

The random variable $S - t'$ is \mathcal{F}_τ -measurable, so with positive probability the conditional law of $\tau' - \tau$ given \mathcal{F}_τ has an atom at $S - t'$. By Lemma 2.1, the conditional law of $\tau' - \tau$ given \mathcal{F}_τ is that of the minimum of finitely many independent exponential random variables, so a.s. this conditional law is non-atomic. This contradiction completes the proof. \square

The following is the main input in the proof of Proposition 2.7.

Lemma 2.10. *Suppose we are in the setting of Proposition 2.7. For $i \in \{1, 2\}$, let σ_i be the smallest $t > \tau$ for which $\#(\mathcal{V}(A_t \cap \Gamma_i)) = \infty$. Also let E_i be the event that there is no path in $A_{\sigma_i} \setminus A_\tau$ which contains an edge in Γ_i and an edge in $\partial\mathcal{E}(\Gamma_i)$. Then*

$$\mathbb{P}(\sigma_1 = \sigma_2 = \tau_\infty, E_1 \cap E_2 \mid \mathcal{F}_\tau) = 0. \quad (2.11)$$

Proof. We first argue that the random variables $\sigma_1 \mathbb{1}_{E_1}$ and $\sigma_2 \mathbb{1}_{E_2}$ are conditionally independent given \mathcal{F}_τ . To see this, define the random variables \widehat{X}_e^τ for $e \in \mathcal{E}(G) \setminus \mathcal{E}(A_\tau)$ as in Definition 2.2, so that the conditional law of the \widehat{X}_e^τ 's given \mathcal{F}_τ is that of a collection of iid exponential random variables with parameters $\text{wt}(e)$. Since Γ_1 and Γ_2 lie at graph distance at least 3 from one another, the sets $\mathcal{E}(\Gamma_1) \cup \partial\mathcal{E}(\Gamma_1)$ and $\mathcal{E}(\Gamma_2) \cup \partial\mathcal{E}(\Gamma_2)$ are disjoint. Therefore, the collections of random variables

$$\left\{ \widehat{X}_e^\tau : e \in (\mathcal{E}(\Gamma_i) \cup \partial\mathcal{E}(\Gamma_i)) \setminus \mathcal{E}(A_\tau) \right\} \quad (2.12)$$

for $i \in \{1, 2\}$ are conditionally independent given \mathcal{F}_τ .

For $i \in \{1, 2\}$, let $\widehat{\tau}_i$ be the smallest $t > \tau$ for which the following is true. For infinitely many $v \in \mathcal{V}(\Gamma_i)$, there exists an infinite path η in G from $\partial\mathcal{V}(A_\tau)$ to v which is contained in Γ_i and satisfies

$$\sum_{e \in \eta} \widehat{X}_e^\tau \leq t - \tau. \quad (2.13)$$

Also let $\hat{\sigma}_i$ be the smallest $t > \tau$ for which there exists a finite path η in $\mathcal{E}(G) \setminus \mathcal{E}(A_\tau)$ which contains an edge of Γ_i and an edge in $\partial\mathcal{E}(\Gamma_i)$ and satisfies (2.13). Then $\hat{\tau}_i$ and $\hat{\sigma}_i$ are measurable functions of \mathcal{F}_τ and the collection of random variables (2.12). Furthermore, the event E_i occurs if and only if $\hat{\tau}_i < \hat{\sigma}_i$, in which case $\sigma_i = \hat{\tau}_i$. Hence $\sigma_i \mathbb{1}_{E_i}$ is a measurable function of \mathcal{F}_τ and the collection (2.12). Therefore $\sigma_1 \mathbb{1}_{E_1}$ and $\sigma_2 \mathbb{1}_{E_2}$ are conditionally independent given \mathcal{F}_τ .

Now suppose by way of contradiction that (2.11) is false. Then

$$\mathbb{P}(\sigma_1 \mathbb{1}_{E_1} = \sigma_2 \mathbb{1}_{E_2} = \tau_\infty \mid \mathcal{F}_\tau) > 0.$$

Since $\sigma_1 \mathbb{1}_{E_1}$ and $\sigma_2 \mathbb{1}_{E_2}$ are conditionally independent given \mathcal{F}_τ , Lemma 2.8 implies that we can find a \mathcal{F}_τ -measurable random variable S such that with positive probability,

$$\mathbb{P}(\sigma_1 = \sigma_2 = \tau_\infty = S \mid \mathcal{F}_\tau) > 0.$$

In particular, the conditional law of τ_∞ given \mathcal{F}_τ has an atom with positive probability, which contradicts Lemma 2.9. \square

Proof of Proposition 2.7. For $n \in \mathbb{N}$, let τ^n be the smallest $t \geq \tau$ for which $\mathcal{E}(A_t) \setminus \mathcal{E}(A_\tau)$ contains n edges. Let $\sigma_1^n, \sigma_2^n, E_1^n$, and E_2^n be as in Lemma 2.10 with τ^n in place of τ . Then Lemma 2.10 implies that a.s. the event

$$\{\sigma_1^n = \sigma_2^n = \tau_\infty\} \cap E_1^n \cap E_2^n$$

does not occur for any $n \in \mathbb{N}$. On the other hand, every $e \in \mathcal{E}(A_{\tau_\infty})$ is contained in some $\mathcal{E}(A_{\tau^n})$, so if the event in (2.6) occurs then there is a finite $n_0 \in \mathbb{N}$ such that

$$\mathcal{E}(A_{\tau_\infty} \setminus A_{\tau^{n_0}}) \subset \mathcal{E}(\Gamma_1 \cup \Gamma_2) \quad (2.14)$$

and $\sigma_1^{n_0} = \sigma_2^{n_0} = \infty$. Since Γ_1 and Γ_2 lie at graph distance at least 3 from one another, the condition (2.14) implies that there is no path in $\mathcal{E}(A_{\tau_\infty}) \setminus \mathcal{E}(A_{\tau^{n_0}})$ which contains an edge of $\partial\mathcal{E}(\Gamma_1) \cup \partial\mathcal{E}(\Gamma_2)$. Therefore $E_1^{n_0} \cap E_2^{n_0}$ occurs. Hence the event in (2.6) must have probability zero. \square

3 Estimating passage times via a deterministic metric

In the remainder of this paper we will consider the f -weighted FPP process $\{A_t\}_{t \geq 0}$ on \mathbb{Z}^d started from 0, as described in Section 1.3, the associated filtration $\{\mathcal{F}_t\}_{t \geq 0}$ from Definition 1.3, as well as the metric D from Section 1.4.

Throughout this section we allow a general choice of $\alpha \in \mathbb{R}$ and α -weight function f . In this section, we will prove that the metric D is a good approximation for passage times in our FPP model. We start in Section 3.1 by reviewing some known estimates for standard FPP. In Section 3.2, we prove some basic deterministic estimates for D . We then prove upper and lower bounds for f -weighted FPP passage times in terms of D in Section 3.3. These latter bounds will be the key inputs in the proofs of Theorems 1.7, 1.8, and 1.9 in the subsequent sections.

3.1 Rate of convergence estimates for standard FPP

Recall the standard FPP limit shape \mathbb{A} from Section 1.4 and the fattened standard FPP clusters A_t^F for $t \geq 0$ from (1.4). Estimates for the rate of convergence in (1.5) are obtained in [Kes93] and sharpened in [Ale97]. In particular, [Kes93, Theorem 2] tells us that for each $\zeta \in (0, 1/2)$ and each $t > 0$,

$$\mathbb{P}\left(A_t^F \not\subset \left(1 + t^{-1/2+\zeta}\right)t\mathbb{A}\right) = o_t^\infty(t) \quad (3.1)$$

at a rate depending only ζ and d (here we recall the notation $o_t^\infty(t)$ from Section 1.2.1). Furthermore, the proof of [Ale97, Theorem 3.1] shows that for each $\zeta \in (0, 1/2)$ and $t > 0$

$$\mathbb{P}\left(\left(1 - t^{-1/2+\zeta}\right)t\mathbb{A} \not\subset A_t^F\right) = o_t^\infty(t), \quad (3.2)$$

at a rate depending only ζ and d .

Remark 3.1. It is expected that the error exponent $1/2$ in (3.1) and (3.2) is not optimal. Heuristic arguments and numerical simulations suggest that these estimates should hold with $2/3$ in place of $1/2$ in the case $d = 2$; see [KPZ86] as well as the discussion immediately following [Kes93, Theorem B] and the references therein.¹ If we had such improved error estimates, then we would also obtain better error estimates in Lemmas 3.6 and 3.7, which would lead to better error estimates in Theorem 1.7.

We will sometimes have occasion to apply (3.1) and (3.2) with the scaling of \mathbb{A} , rather than the time t , specified. For this reason, we record the following estimates, which are immediate from (3.1) and (3.2). For each $r > 0$,

$$\mathbb{P}(A_{r-r^{1/2+\zeta}}^F \not\subset r\mathbb{A}) = o_r^\infty(r), \quad \forall \zeta > 0 \quad (3.3)$$

and

$$\mathbb{P}(r\mathbb{A} \not\subset A_{r+r^{1/2+\zeta}}^F) = o_r^\infty(r), \quad \forall \zeta > 0 \quad (3.4)$$

at a rate depending only on ζ and d .

3.2 Estimates for the weighted metric

In this subsection we prove some basic estimates for the metric D of (1.10) which will be used to compare D -distances to f -weighted FPP distances. We first have an upper bound for D -distances in terms of Euclidean distances.

Lemma 3.2. *Let f be as in (1.1). There is a constant $a > 0$, depending only on f , such that for each $z, w \in \mathbb{R}^d \setminus \{0\}$,*

$$|f(z) - f(w)| \leq a(|z|^{\alpha-1} \vee |w|^{\alpha-1})|z - w|.$$

Proof. Let $w' := (|z|/|w|)w$. By Lipschitz continuity of f_0 and α -homogeneity of f ,

$$|f(z) - f(w')| = |z|^\alpha |f_0(z/|z|) - f_0(w'/|z|)| \leq |z|^{\alpha-1} |z - w'| \leq |z|^{\alpha-1} |z - w|.$$

Furthermore, by the mean value theorem

$$|f(w) - f(w')| \leq ||w|^\alpha - |z|^\alpha| \leq (|z|^{\alpha-1} \vee |w|^{\alpha-1})|z - w|.$$

Combining these inequalities proves the lemma. \square

Our next lemma shows that D is comparable to the metric induced by μ (and hence to that induced by any norm on \mathbb{R}^d) when we restrict attention to sets at positive distance from 0 and ∞ .

Lemma 3.3. *Let $z, w \in \mathbb{R}^d$. Then*

$$\phi(z, w) \leq D(z, w) \leq \mu(z - w) \int_0^1 f(tw + (1-t)z)^{-1} dt \quad (3.5)$$

where, with $\bar{\kappa}$ as in (1.3) and $\bar{\rho}$ as in (1.7),

$$\phi(z, w) = \begin{cases} \frac{|z|^{1-\alpha} - (|z| + \bar{\rho}\mu(w-z))^{1-\alpha}}{\bar{\rho}\bar{\kappa}(\alpha-1)}, & \alpha \in [0, \infty) \setminus \{1\} \\ \bar{\rho}^{-1}\bar{\kappa}^{-1} \log\left(\frac{|z| + \bar{\rho}\mu(w-z)}{|z|}\right), & \alpha = 1 \\ \frac{|z|^{1-\alpha} - (|z| - \bar{\rho}\mu(w-z))^{1-\alpha}}{\bar{\rho}\bar{\kappa}(\alpha-1)}, & \alpha < 0. \end{cases}$$

¹We remark that in any dimension, the error exponent in (3.1) and (3.2) is closely related to the so-called *wandering exponent*, which measures the amount by which FPP geodesics deviate from straight lines. See [Cha13] for a formula relating these exponents as well as [AD14] for a simplified proof of this formula.

Proof. To obtain the upper bound in (3.5), let

$$\gamma(t) := \frac{tw + (\mu(w - z) - t)z}{\mu(w - z)}, \quad \forall t \in [0, \mu(w - z)].$$

Then γ is parametrized by μ -length and by a change of variables,

$$\text{len}^D(\gamma) = \int_0^{\mu(w-z)} f(\gamma(t))^{-1} dt = \mu(w - z) \int_0^1 f(tw + (1 - t)z)^{-1} dt.$$

To obtain the lower bound, let $\gamma : [0, T] \rightarrow \mathbb{R}^d$ be a piecewise linear path from z to w parametrized by μ -length. Then $T \geq \mu(w - z)$ and for each $t \in [0, T]$,

$$|z| - \bar{\rho}t \leq |\gamma(t)| \leq |z| + \bar{\rho}t.$$

Hence for each $t \in [0, T]$,

$$f(\gamma(t))^{-1} \geq \begin{cases} \bar{\kappa}^{-1}(|z| + \bar{\rho}t)^{-\alpha}, & \alpha \geq 0 \\ \bar{\kappa}^{-1}(|z| - \bar{\rho}t)^{-\alpha}, & \alpha < 0. \end{cases}$$

If $\alpha \in [0, \infty) \setminus \{1\}$, we thus have

$$\text{len}^D(\gamma) \geq \bar{\kappa}^{-1} \int_0^T (|z| + \bar{\rho}t)^{-\alpha} dt \geq \frac{|z|^{1-\alpha} - (|z| + \bar{\rho}\mu(w - z))^{1-\alpha}}{\bar{\rho}\bar{\kappa}(\alpha - 1)}.$$

This concludes the proof in the case where $\alpha \in [0, \infty) \setminus \{1\}$. Similar arguments apply in the case where $\alpha < 0$ or $\alpha = 1$. \square

3.3 Growth estimates for weighted FPP

In this subsection, we will use the bounds for standard FPP described in Section 3.1 to prove results which show that the metric D of Section 1.4 is a good approximation for passage times in weighted FPP. The intuition behind the estimates of this subsection is as follows. If $v \in \mathbb{Z}^d$, then near v the f -weighted FPP metric T locally looks like the ordinary ($f \equiv 1$) FPP metric, re-scaled by $f(v)$. This, in turn, is well-approximated by the metric induced by the norm $f(v) \cdot \mu$ due to the estimates of Section 3.1. Hence f -weighted FPP distances are comparable to μ -distances, weighted by f , i.e., D -distances.

We first state our main upper bound for f -weighted FPP passage times. Roughly speaking, the estimate says that if τ is a stopping time for the f -weighted FPP filtration $\{\mathcal{F}_t\}_{t \geq 0}$ and $v \in \mathbb{Z}^d \setminus A_\tau$, then it is very unlikely that the amount of time after τ before v is absorbed by the FPP clusters is too much larger than $D(v, A_\tau)$. The reason for the conditions involving $B_{m^\xi}^{|\cdot|}(0) \setminus B_{m^{1/\xi}}^{|\cdot|}(0)$ in the statement of the lemma is to keep us away from 0 and ∞ , so that we can apply the estimates for D in Section 3.2 and so that we only need to consider polynomially many points (which is important when we apply a union bound to events with probability $o_m^\infty(m)$).

Lemma 3.4 (Upper bound for passage times). *Fix $\xi > 1$. Let τ be a stopping time for $\{\mathcal{F}_t\}_{t \geq 0}$ and let $m \in \mathbb{N}$. For $z, w \in B_{m^\xi}^{|\cdot|}(0) \setminus B_{m^{1/\xi}}^{|\cdot|}(0)$, let*

$$\tilde{D}_m(z, w) := \inf \left\{ \text{len}^D(\gamma) : \gamma \text{ is a piecewise linear path from } z \text{ to } w \text{ in } B_{m^\xi}^{|\cdot|}(0) \setminus B_{m^{1/\xi}}^{|\cdot|}(0) \right\}. \quad (3.6)$$

Also let

$$V_\tau^m := \left(\mathbb{Z}^d \cap \left(B_{m^\xi}^{|\cdot|}(0) \setminus B_{m^{1/\xi}}^{|\cdot|}(0) \right) \right) \setminus \mathcal{V}(A_\tau).$$

Then for each $\beta \in (0, 1/(3\xi))$,

$$\mathbb{P} \left(T(0, v) - \tau \leq (1 + m^{-\beta}) \tilde{D}_m(v, A_\tau) + \frac{m^{1/\xi - \beta}}{|v|^\alpha}, \forall v \in V_\tau^m \mid \mathcal{F}_\tau \right) = 1 - o_m^\infty(m)$$

at a deterministic rate depending only on β and ξ .

We note that the metric \tilde{D}_m of (3.6) is defined in the same manner as D but with a smaller set of allowed paths. Hence $\tilde{D}_m \geq D$. However, if $z, w \in \mathbb{R}^d$ and there is a D -geodesic from z to w which does not enter $B_{m\xi}^{|\cdot|}(0) \setminus B_{m^{1/\xi}}^{|\cdot|}(0)$, the $\tilde{D}_m(z, w) = D(z, w)$.

The following is our main lower bound for f -weighted FPP passage times, which says that (roughly speaking) the amount of time after a stopping time τ before a vertex $v \in \mathbb{Z}^d \setminus A_\tau$ is absorbed is very unlikely to be much larger than $D(v, A_\tau)$.

Lemma 3.5 (Lower bound for passage times). *Fix $\xi > 1$. Let τ be a stopping time for $\{\mathcal{F}_t\}_{t \geq 0}$ with $\tau < \infty$ a.s. and let $m \in \mathbb{N}$. Let \hat{V}_τ^m be the set of $v \in \mathbb{Z}^d \setminus \mathcal{V}(A_\tau)$ such that the FPP geodesic η_v connecting 0 to v satisfies $\eta_v \setminus \mathcal{E}(A_\tau) \subset B_{m\xi}^{|\cdot|}(0) \setminus B_{m^{1/\xi}}^{|\cdot|}(0)$. For $v \in \hat{V}_\tau^m$, let u_v be the last vertex of $\mathcal{V}(A_\tau)$ hit by the geodesic η_v . Let D be the metric (1.10). For each $\beta \in (0, 1/(3\xi))$,*

$$\mathbb{P} \left(T(0, v) - \tau \geq (1 - m^{-\beta}) D(u_v, v) - \frac{m^{1/\xi - \beta}}{|v|^\alpha}, \forall v \in \hat{V}_\tau^m \mid \mathcal{F}_\tau \right) = 1 - o_m^\infty(m)$$

at a deterministic rate depending only on β and ξ .

We will first prove our upper bound for passage times, Lemma 3.4. The following lemma tells us how long it takes for the f -weighted FPP clusters to absorb a μ -ball centered at a vertex in $\partial\mathcal{V}(A_\tau)$. It will be used to prove upper bounds for f -weighted FPP passage times in terms of the metric D by, roughly speaking, considering a piecewise linear path covered by many small balls.

Lemma 3.6. *Let $x \in \partial\mathbb{D}$ and $m \in \mathbb{N}$. Let τ be a stopping time for $\{\mathcal{F}_t\}_{t \geq 0}$. Let v_* be a vertex in $\partial\mathcal{V}(A_\tau)$, chosen in some \mathcal{F}_τ -measurable manner. Let σ be the smallest $t > 0$ for which each element of $\mathbb{Z}^d \cap B_m^\mu(v_*)$ belongs to $\mathcal{V}(A_t)$. Also let $\theta > 1/2$. There is a constant $a > 0$, depending only on f , μ , and θ , such that the following is true. On the event $\{f(v_*) \geq 2a|v_*|^{\alpha-1}m\}$, we have*

$$\mathbb{P} \left(\sigma - \tau \leq \frac{m + m^\theta}{f(v_*) - a|v_*|^{\alpha-1}m} \mid \mathcal{F}_\tau \right) = 1 - o_m^\infty(m) \quad (3.7)$$

at a deterministic rate (here we recall the notation $o_m^\infty(m)$ from Section 1.2.1).

Proof. Define the normalized edge passage times \bar{X}_e^τ for $e \in \mathcal{E}(\mathbb{Z}^d)$ and the corresponding clusters $\bar{A}_{v,s}^\tau$ for $s \geq 0$ and $v \in \mathbb{Z}^d$ as in Definition 2.5.

Let t_* be the smallest $s > 0$ for which $\mathbb{Z}^d \cap B_m^\mu(v_*) \subset \mathcal{V}(\bar{A}_{v_*,s}^\tau)$. By definition of FPP, for each $v \in \mathbb{Z}^d \cap B_m^\mu(v_*)$, there is a simple path η_v in \bar{A}_{v_*,t_*}^τ with $\bar{T}^\tau(\eta_v) \leq t_*$ which connects v_* to v . By assertion 1 of Lemma 2.6, if we let $M_v := \min_{e \in \eta_v} f(m_e)$, where m_e is the midpoint of the edge $e \in \mathcal{E}(\mathbb{Z}^d)$, then

$$v \in \mathcal{V} \left(A_{\tau + M_v^{-1}t_*} \right). \quad (3.8)$$

Hence it suffices to prove an upper bound for t_* and a lower bound for $\min_{v \in (\mathbb{Z}^d \cap B_m^\mu(v_*)) \setminus \mathcal{V}(A_\tau)} M_v$.

To this end, let $v \in (\mathbb{Z}^d \cap B_m^\mu(v_*)) \setminus A_\tau$ be chosen in a \mathcal{F}_τ -measurable manner. By Lemma 2.6, the conditional law given \mathcal{F}_τ of $\{\bar{A}_{v_*,s}^\tau\}_{s \geq 0}$ is that of a standard FPP process starting from v_* (i.e. with weight 1 at each edge). If $s > 0$ and $t_* > s$, then $B_m^\mu(v_*) \cap \mathbb{Z}^d \not\subset \mathcal{V}(\bar{A}_{v_*,s}^\tau)$. By the rate of convergence bound (3.4),

$$\mathbb{P} (t_* > m + m^\theta \mid \mathcal{F}_\tau) = o_m^\infty(m). \quad (3.9)$$

By (3.1),

$$\mathbb{P} \left(\sup \left\{ |u - v_*| : u \in \mathcal{V} \left(\bar{A}_{v_*, m + m^\theta}^\tau \right) \right\} > \bar{\rho}m + 2m^\theta \mid \mathcal{F}_\tau \right) = o_m^\infty(m). \quad (3.10)$$

By Lemma 3.2, we can find a constant $a > 0$ as in the statement of the lemma such that whenever $|u - v_*| \leq \bar{\rho}m + 2m^\theta$ and $f(v_*) \geq 2a|v_*|^{\alpha-1}m$,

$$f(u) \geq f(v_*) - a|v_*|^{\alpha-1}m. \quad (3.11)$$

By combining (3.9) and (3.10), we obtain that if $f(v_*) \geq 2a|v_*|^{\alpha-1}m$, then with conditional probability $1 - o_m^\infty(m)$ given \mathcal{F}_τ , we have $t_* \leq m + m^\theta$ and $M_v \geq f(v_*) - a|v_*|^{\alpha-1}m$. By combining this with (3.8) and a union bound over all $v \in \mathbb{Z}^d \cap B_m^\mu(v_*)$ we conclude. \square

Proof of Lemma 3.4. Let $v_* \in V_\tau^m$ be chosen in an \mathcal{F}_τ -measurable manner. By definition of \tilde{D}_m , there is a piecewise linear path γ contained in $B_{m^\xi}^{|\cdot|}(0) \setminus B_{m^{1/\xi}}^{|\cdot|}(0)$ which connects some element of $\mathcal{V}(A_\tau)$ to v and satisfies

$$\text{len}^D(\gamma) \leq (1 + m^{-100\xi}) \tilde{D}_m(v, A_\tau).$$

Choose some such path γ in a \mathcal{F}_τ -measurable manner. We set $T = \text{len}^\mu(\gamma)$ and take γ to be parametrized by μ -length (Definition 1.5). It follows from Lemma 3.3 that T is at most a constant times some power of m (the constant and the exponent depend only on f and ξ). We will show using Lemma 3.6 that (roughly speaking) the amount of time it takes the FPP clusters to traverse γ is not too much longer than the D -length of γ .

We first construct a modified version of γ , which we call $\tilde{\gamma}$, whose D -length is not too much larger than that of γ and for which FPP passage times between points of $\tilde{\gamma}$ are easier to estimate. To this end, fix $\zeta \in (0, 1/\xi)$, to be chosen later. Let s_0 be the last time $s \in [0, 1]$ for which $\gamma(s) \in A_\tau^F$ (as defined in (1.4)). Let v_0 be the element of $\mathcal{V}(A_\tau)$ closest to $\gamma(s_0)$ in the Euclidean norm. Inductively, if $k \in \mathbb{N}$ and s_{k-1} and v_{k-1} have been defined, let s_k be the first time s after s_{k-1} for which $\gamma(s) \in \partial B_{m^\zeta}^\mu(\gamma(s_{k-1}))$, or $s_k = 1$ if no such s exists. Also let v_k be the element of \mathbb{Z}^d closest to $\gamma(s_k)$ in the Euclidean norm. Let k_* be the smallest $k \in \mathbb{N}$ for which $s_k = 1$ (and hence $v_k = v_{k_*}$). Let $\tilde{\gamma}$ be the piecewise linear path which is the concatenation of the line segments $[v_{k-1}, v_k]$ for $k \in [1, k_*]_{\mathbb{Z}}$.

We will now estimate $\text{len}^D(\tilde{\gamma})$. For each $k \in [1, k_*]_{\mathbb{Z}}$,

$$\mu(v_k - v_{k-1}) \leq \mu(\gamma(s_k) - \gamma(s_{k-1})) + C \leq s_k - s_{k-1} + C, \quad (3.12)$$

where here $C > 0$ is a deterministic constant depending only on μ . Note that in the second inequality we have used that straight lines are geodesics for the metric induced by the norm μ and that γ is parametrized by μ -length. The sets $\gamma([s_{k-1}, s_k])$ and $[v_{k-1}, v_k]$ are each contained in the Euclidean ball $B_{\bar{\rho}m^\zeta + C}^{|\cdot|}(v_{k-1})$, where $\bar{\rho}$ is the constant from Definition 1.4. By Lemma 3.2, for each $y \in \gamma([s_{k-1}, s_k]) \cup [v_{k-1}, v_k]$, we have $|f(v_{k-1}) - f(y)| \leq |v_{k-1}|^{\alpha-1}m^\zeta$ with the implicit constant depending only on f . Note that here we have used that $|y| \geq m^{1/\xi} - \bar{\rho}m^\zeta$ and that $\zeta < 1/\xi$. Therefore,

$$\frac{\sup_{y \in [v_{k-1}, v_k]} f(y)^{-1}}{\inf_{y \in \gamma([s_{k-1}, s_k])} f(y)^{-1}} \leq \frac{f(v_{k-1}) + |v_{k-1}|^{\alpha-1}m^\zeta}{f(v_{k-1}) - |v_{k-1}|^{\alpha-1}m^\zeta} \leq 1 + O_m(m^{\zeta-1/\xi}), \quad (3.13)$$

at deterministic rate depending only on μ and f . By (3.12) and (3.13) we find that

$$\begin{aligned} \text{len}^D(\tilde{\gamma}) &\leq \left(1 + O_m(m^{\zeta-1/\xi}) + O_m(m^{-\zeta})\right) \text{len}^D(\gamma) + O_m(m^\zeta) |v_*|^{-\alpha} \\ &\leq \left(1 + O_m(m^{\zeta-1/\xi}) + O_m(m^{-\zeta})\right) \tilde{D}_m(v_*, A_\tau) + O_m(m^\zeta) |v_*|^{-\alpha} \end{aligned} \quad (3.14)$$

where here the last term $O_m(m^\zeta) |v_*|^{-\alpha}$ comes from the final segment $[v_{k_*-1}, v_{k_*}]$.

It remains to estimate the amount of time it takes for the clusters A_t to traverse the marked vertices on the path $\tilde{\gamma}$. Let $t_0 = \tau$ and for $k \in [1, k_*]_{\mathbb{Z}}$, let t_k be the smallest $t > 0$ for which $v_k \in \mathcal{V}(A_t)$. By Lemma 3.6, for any $\theta \in (1/2, 1)$, there is a constant $a > 0$ such that it holds except on an event of conditional probability $o_m^\infty(m)$ given \mathcal{F}_τ that

$$t_k - t_{k-1} \leq \frac{m^\zeta + m^{\zeta\theta}}{f(v_{k-1}) - a|v_{k-1}|^{\alpha-1}m^\zeta}, \quad \forall k \in [1, k_*]_{\mathbb{Z}}. \quad (3.15)$$

We have $\mu(v_k - v_{k-1}) = m^\zeta + O_m(1)$ for $k \in [1, k_* - 1]_{\mathbb{Z}}$ and

$$\sup_{y \in [v_{k-1}, v_k]} f(y)^{-1} \geq (f(v_{k-1}) + |v_{k-1}|^{\alpha-1}m^\zeta)^{-1}.$$

Hence (3.15) implies that

$$\frac{t_k - t_{k-1}}{\mu(v_k - v_{k-1}) \sup_{y \in [v_{k-1}, v_k]} f(y)^{-1}} \leq 1 + O_m(m^{-\zeta(1-\theta)}) + O_m(m^{\zeta-1/\xi}). \quad (3.16)$$

If we choose $\zeta = \xi^{-1}(2 - \theta)^{-1}$, then $-\zeta(1 - \theta) = \zeta - 1/\xi$ and $-\zeta < \zeta - 1/\xi$. If we are given $\beta \in (0, 1/(3\xi))$ and we choose θ sufficiently close to $1/2$, then we can arrange that the sum of the error terms on the right

side of (3.16) is $O_m(m^{-\beta})$, the sum of the error terms being multiplied by $\tilde{D}_m(v_*, A_\tau)$ on the right side of (3.14) is at most $O_m(m^{-\beta})$, and $\zeta < 1/\xi - \beta$. By (3.15), we also have

$$t_{k_*} - t_{k_*-1} = O_m(m^\zeta) |v_*|^{-\alpha} = O_m(m^{1/\xi-\beta}) |v_*|^{-\alpha}.$$

By summing over all $k \in [1, k_*]_{\mathbb{Z}}$, we find that except on an event of conditional probability $o_m^\infty(m)$ given \mathcal{F}_τ ,

$$T(0, v_*) - \tau \leq (1 + O_m(m^{-\beta})) \text{len}^D(\tilde{\gamma}) + O_m(m^{1/\xi-\beta}) |v_*|^{-\alpha}.$$

We conclude by combining this with (3.14), applying the union bound, and slightly increasing β . \square

We next prove our lower bound for FPP passage times. For the proof of Lemma 3.5, we need the following lemma to help us translate the estimates of Section 3.1 to a lower bound for weighted FPP passage times in terms of D . Roughly speaking, the lemma tells us that if τ is a stopping time for the f -weighted FPP clusters, then it is very unlikely that $A_{\tau+s}$ contains an edge whose μ -distance to A_τ is too large.

Lemma 3.7. *Let τ be a stopping time for $\{\mathcal{F}_t\}_{t \geq 0}$. Also let $v_* \in \partial\mathcal{V}(A_\tau)$ be chosen in a \mathcal{F}_τ -measurable manner. Fix $\theta \in (1/2, 1)$ and for $s > 0$ and $R > 0$, let $F_\tau^{s,R}(v_*)$ be the event that there is an edge $e_* \in \mathcal{E}(\mathbb{Z}^d) \setminus \mathcal{E}(A_\tau)$ such that the following is true.*

1. $e_* \in \mathcal{E}(A_{\tau+s})$.
2. Let η_{e_*} be the FPP geodesic from 0 to e_* . Then $f(z) \leq R$ for each $e \in \eta_{e_*}$ and $z \in e$.
3. v_* is the last vertex in $\partial\mathcal{V}(A_\tau)$ crossed by η_{e_*} .
4. $e_* \notin B_{Rs+(Rs)^\theta}^\mu(v_*)$.

Then for each $p > 0$,

$$\mathbb{P}(F_\tau^{s,R}(v_*) | \mathcal{F}_\tau) \leq (Rs)^{-p},$$

with the implicit constant depending only on p, μ , and f .

In the statement of Lemma 3.7, one should think of R as being large (if $\alpha < 0$), small (if $\alpha > 0$), or of constant order (if $\alpha = 0$) and s as being much larger than $1/R$, so that $Rs \gg 1$.

Proof of Lemma 3.7. Define the random variables \overline{X}_e^τ for $e \in \mathcal{E}(\mathbb{Z}^d)$ and the clusters $\overline{A}_{v_*,s}^\tau$ for $s \geq 0$ and $v \in \mathbb{Z}^d$ as in Definition 2.5.

Suppose the event $F_\tau^{s,R}(v_*)$ occurs. Let $e_* \in \mathcal{E}(\mathbb{Z}^d) \setminus \mathcal{E}(A_\tau)$ and η_{e_*} be as in the definition of $F_\tau^{s,R}(v_*)$. By conditions 1 and 2 in the definition of $F_\tau^{s,R}(v_*)$ together with assertion 1 of Lemma 2.6, $e_* \in \mathcal{E}(\overline{A}_{v_*,Rs}^\tau)$. By condition 4 in the definition of $F_\tau^{s,R}$, we therefore have

$$F_\tau^{s,R}(v_*) \subset \left\{ \overline{A}_{v_*,Rs}^\tau \not\subset v_* + (Rs + (Rs)^\theta) \mathbb{A} \right\}.$$

Since the conditional law of $\overline{A}_{v_*,Rs}^\tau$ given \mathcal{F}_τ is that of a time- Rs standard FPP cluster based at v_* (Lemma 2.6) we deduce the statement of the lemma from Kesten's upper bound (3.1). \square

Proof of Lemma 3.5. The basic idea of the proof is to use an FPP geodesic to construct a piecewise linear path from u_v to v whose D -length is bounded above.

We first define an event on which we have lower bounds for certain FPP passage times, building on the event of Lemma 3.7. Fix $\hat{\theta} \in (0, 1/2)$ and $\theta \in (1/2, 1 - \hat{\theta})$. For $t > 0$, $s > 0$, $R > 0$, and $v \in \mathcal{V}(A_t)$, define the event $F_t^{s,R}(v)$ as in Lemma 3.7 with the above choice of θ . Also fix $\zeta \in (0, 1/\xi)$ and a constant $a > 0$ (to be chosen later, in a manner depending only on μ and f). For $v \in \mathbb{Z}^d$, let

$$E^m(v) := F_{T(0,v) \vee \tau}^{s,R}(v)^c \quad \text{for} \quad s = \frac{(1 - m^{-\hat{\theta}\zeta}) m^\zeta}{f(v) + a|v|^{\alpha-1} m^\zeta} \quad \text{and} \quad R = f(v) + a|v|^{\alpha-1} m^\zeta.$$

Lemma 3.7 and the union bound imply that the event

$$E_*^m := \{E^m(v), \forall v \in (\partial\mathcal{V}(A_\tau) \cup (\mathbb{Z}^d \setminus \mathcal{V}(A_\tau))) \cap B_{2m^\epsilon}(0)\}$$

has conditional probability $1 - o_m^\infty(m)$ given \mathcal{F}_τ .

Suppose now that E_*^m occurs. Let $v_* \in \widehat{V}_\tau^m$ be chosen in some \mathcal{F}_τ -measurable manner. Let η_{v_*} be the FPP geodesic from 0 to v_* , so that by definition of \widehat{V}_τ^m , we have $\eta_{v_*} \subset B_{m^\epsilon}^{|\cdot|}(0)$. Let $v_0 = u_{v_*}$ be as in the statement of the lemma. Inductively, for $k \in \mathbb{N}$ let v_k be the first vertex in $\mathcal{V}(\mathbb{Z}^d)$ hit by η_{v_*} after it hits v_{k-1} which does not lie in $B_{m^\epsilon}^\mu(v_{k-1})$; or $v_k = v_*$ if no such vertex exists. Let k_* be the smallest $k \in \mathbb{N}$ for which $v_k = v_*$. For $k \in [1, k_*]_\mathbb{Z}$, let η_k be the segment of η between v_{k-1} and v_k . By definition, each η_k is contained in $A_{T(0, v_k)} \cap B_{\bar{\rho}m^\epsilon + C}^{|\cdot|}(v_{k-1})$ for an appropriate $C > 0$ depending only on μ . By Lemma 3.2, we can find a constant $a > 0$, depending only on μ and f , such that

$$f(z) \leq f(v_{k-1}) + a|v_{k-1}|^{\alpha-1}m^\zeta, \quad \forall e \in \eta_k \text{ and } z \in e.$$

We henceforth take this choice of a in the definition of the events $E^m(v)$ above.

By definition of $E^m(v_{k-1})$ (c.f. the definition of the event from Lemma 3.7), we have for large enough m

$$T(0, v_k) - T(0, v_{k-1}) \vee \tau \geq \frac{(1 - m^{-\widehat{\theta}\zeta})m^\zeta}{f(v_{k-1}) + a|v_{k-1}|^{\alpha-1}m^\zeta}, \quad \forall k \in [1, k_* - 1]_\mathbb{Z}.$$

Hence for large enough m ,

$$T(0, v_*) - \tau \geq \sum_{k=1}^{k_*-1} \frac{(1 - m^{-\widehat{\theta}\zeta})m^\zeta}{f(v_{k-1}) + a|v_{k-1}|^{\alpha-1}m^\zeta}. \quad (3.17)$$

Let γ be the concatenation of the line segments $[v_{k-1}, v_k]$ for $k \in [1, k_*]_\mathbb{Z}$. Then γ is a piecewise linear path from u_{v_*} to v_* . Furthermore, each point of each segment $[v_{k-1}, v_k]$ lies within Euclidean distance $\bar{\rho}m^\zeta + C$ of v_{k-1} (with $\bar{\rho}$ the constant from Definition 1.4) so by Lemma 3.2,

$$\sup_{y \in [v_{k-1}, v_k]} f(y)^{-1} \leq (f(v_{k-1}) - a'|v_{k-1}|^{\alpha-1}m^\zeta)^{-1}$$

for appropriate deterministic $a' > 0$ depending only on μ and f . Hence the definition (1.10) of D implies that

$$\sum_{k=1}^{k_*} \frac{\mu(v_k - v_{k-1})}{f(v_{k-1}) - a'|v_{k-1}|^{\alpha-1}m^\zeta} \geq D(u_{v_*}, v_*). \quad (3.18)$$

We have $\mu(v_k - v_{k-1}) \leq m^\zeta + O_m(1)$ for $k \in [1, k_*]_\mathbb{Z}$ and

$$\frac{f(v_{k-1}) - a'|v_{k-1}|^{\alpha-1}m^\zeta}{f(v_{k-1}) + a|v_{k-1}|^{\alpha-1}m^\zeta} = 1 - O_m(m^{\zeta-1/\xi}),$$

at a deterministic rate which does not depend on the particular choices of v_* , k , or the realization of our random variables. Note that here we use that each v_k belongs to $B_{m^\epsilon}^{|\cdot|}(0) \setminus B_{m^{1/\epsilon}}^{|\cdot|}(0)$. Hence

$$\frac{m^\zeta}{f(v_{k-1}) + a|v_{k-1}|^{\alpha-1}m^\zeta} \geq \frac{(1 - O_m(m^{\zeta-1/\xi}))\mu(v_k - v_{k-1})}{f(v_{k-1}) - a'|v_{k-1}|^{\alpha-1}m^\zeta}, \quad \forall k \in [1, k_* - 1]_\mathbb{Z}$$

and

$$\frac{\mu(v_* - v_{k_*-1})}{f(v_{k_*-1}) - a'|v_{k_*-1}|^{\alpha-1}m^\zeta} = O_m(m^\zeta)|v_*|^{-\alpha}.$$

By combining this with (3.17) and (3.18), we obtain that if $1/\xi - \zeta \geq \widehat{\theta}\zeta$, then

$$T(0, v_*) - \tau \geq (1 - O_m(m^{-\widehat{\theta}\zeta})) D(u_{v_*}, v_*) - O_m(m^\zeta)|v_*|^{-\alpha}. \quad (3.19)$$

Now set $\zeta = \xi^{-1}(1 + \widehat{\theta})^{-1}$, so that $1/\xi - \zeta = \widehat{\theta}\zeta$. If we are given $\beta \in (0, 1/(3\xi))$ and we choose $\widehat{\theta} \in (0, 1/2)$ sufficiently close to $1/2$, then we have $\widehat{\theta}\zeta > \beta$ and $\zeta < 1/\xi - \beta$. Since our choice of $v_* \in \widehat{V}_\tau^m$ was arbitrary, the desired estimate now follows from (3.19). \square

Remark 3.8. The proof of Lemma 3.5 actually yields a slightly stronger but somewhat more complicated version of the statement of the lemma which we will need in Section 5. Suppose we are in the setting of Lemma 3.5. Also let $\mathcal{U} \subset \mathbb{R}^d$ be a deterministic open set and let \mathcal{U}_m be the set of $z \in \mathbb{R}^d$ which lie at Euclidean distance $< m^{1/\xi}$ from \mathcal{U} . Define an internal version of the metric D by

$$\tilde{D}_{\mathcal{U}_m}(z, w) := \inf \left\{ \text{len}^D(\gamma) : \gamma \text{ is a piecewise linear path from } z \text{ to } w \text{ contained in } \mathcal{U}_m \right\}, \quad \forall z, w \in \mathcal{U}_m.$$

Let $\hat{V}_\tau^m(\mathcal{U})$ be the set of vertices $v \in \hat{V}_\tau^m$ such that the corresponding FPP geodesic η_v satisfies $\eta_v \setminus \mathcal{E}(A_\tau) \subset \mathcal{U}$. Then

$$\mathbb{P} \left(T(0, v) - \tau \geq (1 - m^{-\beta}) \tilde{D}_{\mathcal{U}_m}(u_v, v) - \frac{m^{1/\xi - \beta}}{|v|^\alpha}, \forall v \in \hat{V}_\tau^m(\mathcal{U}) \mid \mathcal{F}_\tau \right) = 1 - o_m^\infty(m).$$

Indeed, this follows from the proof of Lemma 3.5 upon noting that, with γ the piecewise linear path defined in the proof, for large enough m (how large is deterministic and depends only on f, ξ , and β) we have $\gamma \subset \mathcal{U}_m$ for each $v_* \in \hat{V}_\tau^m(\mathcal{U})$. Therefore, the estimate (3.18) holds with $\tilde{D}_{\mathcal{U}_m}$ in place of D .

4 Proof of limit shape and covering results

4.1 Proof of Theorem 1.7

In this subsection we will use the estimates of Section 3.3 to prove Theorem 1.7. For the proof, we use the setup of Theorem 1.7, so in particular we always assume $\alpha < 1$ and we let $\{A_t\}_{t \geq 0}$ be the f -weighted FPP clusters and $\{\mathcal{F}_t\}_{t \geq 0}$ be the associated filtration, as in Definition 1.3.

We also introduce the following additional notation. For $r \geq 0$, let

$$\tau_r := \inf \{t \geq 0 : A_t \not\subset B_r^D(0)\}. \quad (4.1)$$

For $0 \leq r' \leq r$, let

$$G_{r,r'} := \{B_{r'}^D(0) \cap \mathbb{Z}^d \subset \mathcal{V}(A_{\tau_r})\} \quad (4.2)$$

be the event that the f -weighted FPP clusters fill in $B_{r'}^D(0)$ before time τ_r . Then τ_r is a $\{\mathcal{F}_t\}_{t \geq 0}$ -stopping time and $G_{r,r'} \in \mathcal{F}_{\tau_r}$.

The basic outline of the proof of Theorem 1.7 is as follows. In Lemma 4.2, we will use the estimates of Section 3.3 to prove that if $\tilde{n} \geq n$ with $\tilde{n} \asymp n$, then with high conditional probability given \mathcal{F}_{τ_n} , it holds that $\tau_{\tilde{n}} - \tau_n$ is not too far from $\tilde{n} - n$. In Lemma 4.3, we will use Lemma 3.5 to show that if $m \leq n \leq \tilde{n}$ with $m \asymp n \asymp \tilde{n}$, then on $G_{n,m}$ it holds with high conditional probability given \mathcal{F}_{τ_n} that the event $G_{\tilde{n}, \tilde{m}}$ occurs for \tilde{m} slightly smaller than $m + \tilde{n} - n$. Both of these two lemmas are proven using the estimates of Section 3.3. Together with a straightforward induction argument, these lemmas imply that if $a > 0$ is fixed and $G_{n,an}$ occurs for large enough n , then the event of Theorem 1.7 occurs with high probability. To complete the proof of Theorem 1.7, we still need to show that for an appropriate choice of constant $a > 0$, we have $\mathbb{P}(G_{n,an}) = 1 - o_n^\infty(n)$. This is accomplished in Lemma 4.4.

We first record the following convenient fact, which is an immediate consequence of (1.12).

Lemma 4.1. *There is a constant $c > 1$, depending only on μ and f such that*

$$c^{-1}r^{\frac{1}{1-\alpha}}\mathbb{D} \subset r\mathbb{B} \subset cr^{\frac{1}{1-\alpha}}\mathbb{D}, \quad \forall r > 0. \quad (4.3)$$

We now use Lemma 4.1 and the estimates of Section 3.3 to prove some basic estimates for the D -ball exit times τ_n .

Lemma 4.2. *Fix $R \geq 2$, $\delta > 0$, and χ as in (1.13). For $n \in \mathbb{N}$ and $\tilde{n} \in [n + R^{-1}n, Rn]_{\mathbb{Z}}$, we have (in the notation (4.1))*

$$\mathbb{P}(\tau_{\tilde{n}} - \tau_n \geq \tilde{n} - n - \delta n^{1-\chi} \mid \mathcal{F}_{\tau_n}) = 1 - o_n^\infty(n). \quad (4.4)$$

Furthermore, for each $m \in [R^{-1}n, n]$,

$$\mathbb{P}(\tau_{\tilde{n}} - \tau_n \leq \tilde{n} - m + \delta n^{1-\chi} \mid \mathcal{F}_{\tau_n}) \mathbb{1}_{G_{n,m}} = (1 - o_n^\infty(n)) \mathbb{1}_{G_{n,m}}. \quad (4.5)$$

The $o_n^\infty(n)$ errors above are deterministic and independent of the particular choices of \tilde{n} and m , but may depend on α, f , and R .

Proof. First we consider the lower bound (4.4). We first reduce to the case when the realization of A_{τ_n} is as large as possible. Let U_n be the set of $u \in \mathbb{Z}^d \setminus B_n^D(0)$ such that u is incident to a vertex in $B_n^D(0) \cap \mathbb{Z}^d$. For $u \in U_n$, let \mathfrak{A}_u be the subgraph of \mathbb{Z}^d whose vertex set is $(\mathbb{Z}^d \cap B_n^D(0)) \cup \{u\}$ and whose edge set is the set of all edges in $\mathcal{E}(\mathbb{Z}^d)$ which join vertices in its vertex set. Almost surely, the set A_{τ_n} contains exactly one element of U_n and no other elements of $\mathbb{Z}^d \setminus B_n^D(0)$. By Lemma 2.4, the conditional law of $\tau_{\tilde{n}} - \tau_n$ given \mathcal{F}_{τ_n} a.s. stochastically dominates the conditional law of $\tau_{\tilde{n}} - \tau_n$ given $\{A_{\tau_n} = \mathfrak{A}_u\}$ for some $u \in U_n$. Hence to prove (4.4) it suffices to show that

$$\mathbb{P}(\tau_{\tilde{n}} - \tau_n \geq \tilde{n} - n - \delta n^{1-\chi} \mid A_{\tau_n} = \mathfrak{A}_u) = 1 - o_n^\infty(n), \quad (4.6)$$

uniformly over all choices of $u \in U_n$.

To this end, let $u_{\tilde{n}}$ be the (a.s. unique) element of $\mathcal{V}(A_{\tau_{\tilde{n}}}) \setminus B_{\tilde{n}}^D(0)$. By Lemma 4.1,

$$D(u_{\tilde{n}}, \mathfrak{A}_u) \geq \tilde{n} - n + O_n(n^{-\frac{\alpha}{1-\alpha}}), \quad \forall u \in U_n.$$

Furthermore, by Lemma 4.1, on the event $\{A_{\tau_n} = \mathfrak{A}_u\}$ for $u \in U_n$, the FPP geodesic $\eta_{u_{\tilde{n}}}$ from 0 to $u_{\tilde{n}}$ satisfies

$$\eta_{u_{\tilde{n}}} \setminus \mathcal{E}(A_{\tau_n}) \subset B_{2\tilde{n}}^D(0) \setminus B_n^D(0) \subset B_{Cn^{\frac{1}{1-\alpha}}}^{| \cdot |}(0) \setminus B_{C^{-1}n^{\frac{1}{1-\alpha}}}^{| \cdot |}(0)$$

for an appropriate constant $C > 0$, depending only on R, μ , and f . Therefore, Lemma 3.5 (applied with $n^{\frac{1}{1-\alpha}}$ in place of m and ξ slightly larger than 1) implies that for each $\beta \in (0, 1/3)$, the following is true. For each $u \in U_n$, it holds except on an event of conditional probability $1 - o_n^\infty(n)$ given $\{A_{\tau_n} = \mathfrak{A}_u\}$ (at a rate independent from u) that

$$\tau_{\tilde{n}} - \tau_n = T(0, u_{\tilde{n}}) - \tau_n \geq \left(1 - n^{-\frac{\beta}{1-\alpha}}\right) (\tilde{n} - n - O_n(n^{-\frac{\alpha}{1-\alpha}})) - O_n\left(n^{\frac{1-\beta-\alpha}{1-\alpha}}\right) \geq \tilde{n} - n - O_n\left(n^{\frac{1-\beta-\alpha}{1-\alpha}}\right), \quad (4.7)$$

provided β is chosen sufficiently close to $1/3$. If we choose β sufficiently close to $1/3$, then for large enough n the error term on the right side of (4.7) is smaller than $\delta n^{1-\chi}$. This proves (4.6).

Now we turn our attention to the upper bound (4.5). To this end, suppose $G_{n,m}$ occurs. We can choose $v \in \mathbb{Z}^d \cap B_m^D(0)$ and $v' \in \mathbb{Z}^d \setminus B_{\tilde{n}}^D(0)$ (in some \mathcal{F}_{τ_n} -measurable manner) in such a way that

$$D(v, v') \leq \tilde{n} - m + O_n(n^{-\frac{\alpha}{1-\alpha}}). \quad (4.8)$$

Since $G_{n,m}$ occurs, $v \in \mathcal{V}(A_{\tau_n})$. We remark that for an arbitrary choice of $v \in \partial B_m^D(0)$, there need not exist $v' \in \mathbb{Z}^d \setminus \partial B_{\tilde{n}}^D(0)$ for which (4.8) holds; this is why we need to assume that $G_{n,m}$ occurs in (4.4).

By definition, $T(0, v') \leq \tau_{\tilde{n}}$. For each $\epsilon > 0$, we can find a piecewise linear path γ connecting v to v' with

$$\text{len}^D(\gamma) \leq D(v, v') + \epsilon \leq \tilde{n} - m + O_n(n^{-\frac{\alpha}{1-\alpha}}) + \epsilon.$$

We observe that for small enough ϵ and large enough n , this path γ cannot enter $B_{m/2}^D(0)$ or exit $B_{2\tilde{n}}^D(0)$. Indeed, if this were the case then we would have

$$\text{len}^D(\gamma) \geq \tilde{n} - m + \frac{m}{2} \wedge \tilde{n}.$$

It therefore follows from Lemma 3.4 that for each $\beta \in (0, 1/3)$, it holds except on an event of conditional probability $1 - o_n^\infty(n)$ given \mathcal{F}_{τ_n} that

$$T(0, v') - \tau_n \leq \left(1 + n^{-\frac{\beta}{1-\alpha}}\right) (\tilde{n} - m + O_n(n^{-\frac{\alpha}{1-\alpha}})) + O_n\left(n^{\frac{1-\beta-\alpha}{1-\alpha}}\right) = \tilde{n} - m + O_n\left(n^{\frac{1-\beta-\alpha}{1-\alpha}}\right). \quad (4.9)$$

By choosing β sufficiently close to $1/3$, we conclude. \square

Our next lemma (plus an induction argument) will eventually tell us that if for some $n \in \mathbb{N}$, $\mathcal{V}_{A_{\tau_n}}$ contains $B_m^D(0) \cap \mathbb{Z}^d$ for m at least a constant times n , then with high probability the same is in fact true for all sufficiently large $n \in \mathbb{N}$.

Lemma 4.3. Fix $R \geq 2$, $\delta > 0$, and χ as in (1.13). For $n \in \mathbb{N}$, $m \in [R^{-1}n, n]_{\mathbb{Z}}$, and $\tilde{n} \in [n + R^{-1}n, Rn]_{\mathbb{Z}}$, set $\tilde{m} := m + \tilde{n} - n - \delta n^{1-\chi}$. Then in the notation (4.2),

$$\mathbb{P}(G_{\tilde{n}, \tilde{m}} \mid \mathcal{F}_{\tau_n}) \mathbb{1}_{G_{n,m}} \geq (1 - o_n^\infty(n)) \mathbb{1}_{G_{n,m}},$$

at a deterministic rate independent of the particular choices of m and \tilde{n} but which may depend on α , f , and R .

Proof. Assume $G_{n,m}$ occurs and consider a vertex $v \in (B_m^D(0) \cap \mathbb{Z}^d) \setminus \mathcal{V}(A_{\tau_n})$, chosen in some \mathcal{F}_{τ_n} -measurable manner. Let $A_{\tau_n}^F$ be the fattening of A_{τ_n} , as in (1.4). Since $G_{n,m}$ occurs, $B_m^D(0) \subset A_{\tau_n}^F$ so

$$D(v, A_{\tau_n}^F) \leq \tilde{m} - m. \quad (4.10)$$

By definition of D , for each $\epsilon > 0$ there is a piecewise linear path $\tilde{\gamma}$ connecting some point in $A_{\tau_n}^F$ to v with $\text{len}^D(\tilde{\gamma}) \leq \tilde{m} - m + \epsilon$. By possibly replacing $\tilde{\gamma}$ with its restriction to some interval of times, we can arrange that only the first point of $\tilde{\gamma}$ belongs to $A_{\tau_n}^F$, so that (by definition of $G_{n,m}$) $\tilde{\gamma}$ is disjoint from $B_m^D(0)$. Furthermore, for small enough ϵ , $\tilde{\gamma}$ cannot exit $B_{2\tilde{m}}^D(0)$ for otherwise its D -length would be larger than \tilde{m} . Let u be an element of $\mathcal{V}(A_{\tau_n})$ lying at minimal D -distance from the initial point of $\tilde{\gamma}$, with ties broken in a \mathcal{F}_{τ_n} -measurable manner. By adding a line segment at the beginning of $\tilde{\gamma}$, we obtain for each $\epsilon > 0$ a piecewise linear path γ which connects some $u \in \mathcal{V}(A_{\tau_n})$ to v , is contained in $B_{2\tilde{m}}^D(0) \setminus B_{m - O_n(n^{-\frac{\alpha}{1-\alpha}})}^D(0)$ provided $G_{n,m}$ occurs, and satisfies

$$\text{len}^D(\gamma) \leq \tilde{m} - m + O_n(n^{-\frac{\alpha}{1-\alpha}}) + \epsilon.$$

By Lemmas 3.4 and 4.1, for each $\beta \in (0, 1/3)$, the following is true. Whenever $G_{n,m}$ occurs, it holds except on an event of conditional probability $o_n^\infty(n)$ given \mathcal{F}_{τ_n} that

$$T(0, v) - \tau_n \leq \left(1 + n^{-\frac{\beta}{1-\alpha}}\right) (\tilde{m} - m + O_n(n^{-\frac{\alpha}{1-\alpha}})) + O_n\left(n^{\frac{1-\beta-\alpha}{1-\alpha}}\right) = \tilde{m} - m + O_n\left(n^{\frac{1-\beta-\alpha}{1-\alpha}}\right), \quad (4.11)$$

for every possible choice of $v \in (B_m^D(0) \cap \mathbb{Z}^d) \setminus \mathcal{V}(A_{\tau_n})$. If we choose β sufficiently close to $1/3$ then for large enough n (how large is deterministic and independent of the particular choice of \tilde{n}), the right side of (4.11) is smaller than $\tilde{n} - n - (\delta/2)n^{1-\chi}$. The statement of the lemma now follows from Lemma 4.2. \square

In order to deduce Theorem 1.7 from Lemma 4.3, we need to start with a large $n \in \mathbb{N}$, an integer $m \leq n$ with $m \asymp n$, and a realization of \mathcal{F}_{τ_n} for which $G_{n,m}$ occurs and $\tau_n \asymp n$. Our next lemma will provide such a realization.

Lemma 4.4. There is a constant $a \in (0, 1)$ (independent from n) such that for each $\delta > 0$ and each χ as in (1.13),

$$\mathbb{P}(G_{n,an} \cap \{n - \delta n^{1-\chi} \leq \tau_n \leq a^{-1}n\}) = 1 - o_n^\infty(n), \quad \forall n \in \mathbb{N}.$$

Proof. Fix $R \geq 2$ and $\chi' \in (\chi, 1/3)$. Given $n \in \mathbb{N}$, let $n_0 = \lfloor n^{(1-\chi')/2} \rfloor$. We can select integers $m \in \mathbb{N}$ and $n_0 < n_1 < \dots < n_m = n$ with $n_k \in [(1 + R^{-1})n, Rn]_{\mathbb{Z}}$ for each $k \in [1, m]_{\mathbb{Z}}$ and $m \leq \log n$. By (4.4) of Lemma 4.2 and the union bound, it holds except on an event of probability $1 - o_n^\infty(n)$ that

$$\tau_{n_k} - \tau_{n_{k-1}} \geq n_k - n_{k-1} - n_{k-1}^{1-\chi'}, \quad \forall k \in [1, m]_{\mathbb{Z}}.$$

In this case, $\tau_n \geq n - n_0 - O_n(\log n)n^{1-\chi'}$, which is at least $n - n^{1-\chi}$ for large enough n . Hence

$$\mathbb{P}(\tau_n \geq n - n^{1-\chi}) = 1 - o_n^\infty(n). \quad (4.12)$$

It remains to find an $a \in (0, 1)$ as in the statement of the lemma such that with high probability $G_{n,an}$ occurs and $\tau_n \leq a^{-1}n$. By Lemma 4.1, we can find a $C > 1$ depending only on f such that for each $n \in \mathbb{N}$ and each $v \in B_n^D(0) \cap \mathbb{Z}^d$, we have that v lies at graph distance at most $Cn^{\frac{1}{1-\alpha}}$ and at least $C^{-1}n^{\frac{1}{1-\alpha}}$ from 0. Now fix such a v and let η be a simple path in \mathbb{Z}^d from 0 to v with $|\eta|$ minimal, so $|\eta| \asymp n^{\frac{1}{1-\alpha}}$. Since the function f_0 is bounded above and below by positive constants, there is a constant $c > 0$ depending only

on f such that the law of the random variables $X_{\eta(i)}$ for $i \in [1, |\eta|]_{\mathbb{Z}}$ is that of a collection of independent exponential random variables, each with parameter at least ci^α . Therefore, the law of the passage time $T(\eta)$ is stochastically dominated by the random variable

$$\bar{Y} := \tilde{C} \sum_{i=1}^{\lceil Cn^{\frac{1}{1-\alpha}} \rceil} Y_j$$

where the Y_j 's are independent exponential random variables each with parameter i^α . We have $\mathbb{E}(\bar{Y}) \asymp \int_0^{Cn^{\frac{1}{1-\alpha}}} t^{-\alpha} dt \asymp n$ with the implicit constant depending only on C, \tilde{C} , and α . By elementary tail bounds for sums of exponential random variables (see [Jan14, Theorem 5.1, item (i)]) we can find a constant $a \in (0, 1)$, depending only on f , such that

$$\mathbb{P}(\bar{Y} > a^{-1}n) \leq \begin{cases} \exp(-n), & \alpha \in [0, 1) \\ \exp(-n^{1+\frac{\alpha}{1-\alpha}}), & \alpha < 0. \end{cases}$$

It follows that except on an event of probability $1 - o_n^\infty(n)$, we have $T(\eta) \leq a^{-1}n$, so by a union bound except on an event of probability $1 - o_n^\infty(n)$, $B_n^D(0) \cap \mathbb{Z}^d \subset A_{a^{-1}n}$. In particular, except on an event of probability $o_n^\infty(n)$ we have $\tau_n \leq a^{-1}n$ and by (4.12), $B_{a'n}^D(0) \cap \mathbb{Z}^d \subset A_{\tau_n}$ for a' slightly smaller than a . This proves the statement of the lemma with a' in place of a . \square

Proof of Theorem 1.7. Let a be the constant from Lemma 4.4. Also fix $R > 3 \vee a^{-1}$, let χ be as in (1.13), and let $\delta > 0$ to be chosen later, depending only on R and χ .

For $n, m \in \mathbb{N}$ with $m \leq n$, let τ_n and $G_{n,m}$ be as in (4.1) and (4.2). For $n, n_0 \in \mathbb{N}$ with $n_0 \leq n$ and $k \in \mathbb{N} \cup \{0\}$, let

$$\hat{G}_{n_0,n}^k := G_{n,n-n^{1-\chi}-(1-a)n_0} \cap \{n - n^{1-\chi} \leq \tau_n \leq n + \delta^{-1}n^{1-\chi} + a^{-1}n_0 + k(1-a)n_0\}. \quad (4.13)$$

We first claim that for an appropriate choice of δ , it holds for each $n_0 \in \mathbb{N}$, $n \geq n_0$, and $k \in \mathbb{N}$ that

$$\mathbb{P} \left(\bigcap_{\tilde{n}=\lceil n+R^{-1}n \rceil}^{\lfloor Rn \rfloor} \hat{G}_{n_0,\tilde{n}}^k \mid \mathcal{F}_{\tau_n} \right) \mathbb{1}_{\hat{G}_{n_0,n}^{k-1}} = (1 - o_n^\infty(n)) \mathbb{1}_{\hat{G}_{n_0,n}^k}, \quad (4.14)$$

at a deterministic rate independent from n_0 . To see this, we first apply Lemmas 4.2, 4.3, and the union bound to find that if $n \geq n_0$ and $\hat{G}_{n_0,n}^{k-1}$ occurs, then except on an event of conditional probability $o_n^\infty(n)$ given \mathcal{F}_{τ_n} , it holds for each $\tilde{n} \in [n + R^{-1}n, Rn]_{\mathbb{Z}}$ that

$$\begin{aligned} G_{\tilde{n}, \tilde{n}-(1+\delta)n^{1-\chi}-(1-a)n_0} \text{ occurs and} \\ \tilde{n} - (1+\delta)n^{1-\chi} \leq \tau_{\tilde{n}} \leq \tilde{n} + (\delta^{-1} + 2)n^{1-\chi} + a^{-1}n_0 + k(1-a)n_0. \end{aligned} \quad (4.15)$$

If $\delta > 0$ is chosen sufficiently small depending only on R and χ , then

$$\begin{aligned} \tilde{n}^{1-\chi} &\geq (1 + R^{-1})^{1-\chi} n^{1-\chi} \geq (1 + \delta)n^{1-\chi} \quad \text{and} \\ \delta^{-1}\tilde{n}^{1-\chi} &\geq \delta^{-1}(1 + R^{-1})^{1-\chi} n^{1-\chi} \geq (\delta^{-1} + 2)n^{1-\chi}. \end{aligned}$$

Therefore, $\hat{G}_{n_0,\tilde{n}}^k$ is contained in the event (4.15). This proves (4.14).

By (4.14) and induction, we infer that for each $n_0 \in \mathbb{N}$,

$$\mathbb{P} \left(\bigcap_{n=\lceil (1+R^{-1})n_0 \rceil}^{\infty} \hat{G}_{n_0,n}^{k_n} \mid \mathcal{F}_{\tau_{n_0}} \right) \mathbb{1}_{\hat{G}_{n_0,n_0}^0} = (1 - o_{n_0}^\infty(n_0)) \mathbb{1}_{\hat{G}_{n_0,n_0}^0}, \quad (4.16)$$

where

$$k_n = \left\lfloor \frac{\log n}{\log(1 + R^{-1})} \right\rfloor.$$

By Lemma 4.4,

$$\mathbb{P}\left(\widehat{G}_{n_0, n_0}^0\right) = 1 - o_{n_0}^\infty(n_0). \quad (4.17)$$

By combining (4.16) and (4.17) (the latter applied with $\lfloor (1 + R^{-1})^{-1}n_0 \rfloor$ in place of n_0), we obtain

$$\mathbb{P}\left(\bigcap_{n=\lceil n_0 \rceil}^\infty \widehat{G}_{n_0, n}^{k_n}\right) = 1 - o_{n_0}^\infty(n_0). \quad (4.18)$$

Now suppose that $t_0 > 0$. Set $n_0 = \lfloor 2^{-1}t_0^{(1-\chi)/2} \rfloor$ so that by (4.16), it holds except on an event of probability $o_{t_0}^\infty(t_0)$ that the event $\widehat{G}_{n_0, n}^{k_n+1}$ occurs for each $n \geq n_0$. Let $t \geq t_0$ and let $n \in \mathbb{N}$ be chosen so that $t \in [\tau_n, \tau_{n+1}]$. For large enough values of t_0 , we have $(\log n + 1)n_0 \leq n^{1-\chi}$. By definition of $\widehat{G}_{n_0, n}^{k_n}$,

$$B_{n-O_n(n^{1-\chi})}^D(0) \cap \mathbb{Z}^d \subset \mathcal{V}(A_t) \subset B_{n+O_n(1)}^D(0) \cap \mathbb{Z}^d$$

and

$$n - O_n(n^{1-\chi}) \leq t \leq n + O_n(n^{1-\chi}),$$

with the $O_n(\cdot)$ deterministic and depending only on n , δ , and χ .

Therefore, for an appropriate constant $C' > 0$, depending only on δ and χ ,

$$B_{t-C't^{1-\chi}}^D(0) \cap \mathbb{Z}^d \subset \mathcal{V}(A_t) \subset B_{t+C't^{1-\chi}}^D(0) \cap \mathbb{Z}^d.$$

By (1.12), for a possibly larger constant C' ,

$$(1 - C't^{-\chi})t^{\frac{1}{1-\alpha}}\mathbb{B} \subset A_t^F \subset (1 + C't^{-\chi})t^{\frac{1}{1-\alpha}}\mathbb{B}.$$

We conclude by slightly increasing χ and recalling (4.18). \square

4.2 Proof of Theorem 1.9

Throughout this section, we assume that we are in the setting of Theorem 1.9. In particular, we let σ_r for $r > 0$ be the exit time from the Euclidean ball of radius r centered at 0, as in (1.16).

We note that Lemma 3.3 implies that if $\alpha \geq 1$, then for any $q > r > 1$, then

$$D(q\partial\mathbb{D}, r\partial\mathbb{D}) \geq \begin{cases} \frac{r^{1-\alpha} - q^{1-\alpha}}{\bar{\rho}\bar{\kappa}(\alpha-1)}, & \alpha > 1 \\ \bar{\rho}^{-1}\bar{\kappa}^{-1} \log\left(\frac{q}{r}\right), & \alpha = 1. \end{cases} \quad (4.19)$$

Furthermore, by the definition (1.15) of λ together with Lemma 1.6,

$$\sup_{z, w \in \partial\mathbb{D}} D(z, w) \leq \lambda r^{1-\alpha}, \quad \forall r > 0. \quad (4.20)$$

Roughly speaking, the proof of Theorem 1.9 proceeds as follows. Lemma 3.5 and (4.19) imply that if R is sufficiently large, then for $n \in \mathbb{N}$ it is typically the case that $\sigma_{Rn} - \sigma_n$ is not too much smaller than $(\alpha-1)^{-1}\bar{\rho}^{-1}\bar{\kappa}^{-1}n^{1-\alpha}$. If $(\alpha-1)^{-1}\bar{\rho}^{-1}\bar{\kappa}^{-1} < \lambda$, then Lemma 3.4 and (4.20) imply that with high probability, the clusters A_t absorb every vertex of $\mathbb{Z}^d \cap (B_n^{|\cdot|}(0) \setminus B_{n-1}^{|\cdot|}(0))$ between times σ_n and σ_{Rn} . Sending $n \rightarrow \infty$ concludes the proof. We now proceed with the details.

Lemma 4.5. *Fix $R \geq 2$ and $\beta \in (0, 1/3)$. Suppose $n \in \mathbb{N}$ and $\tilde{n} \in [(1 + R^{-1})n, Rn]_{\mathbb{Z}}$. If $\alpha > 1$, then*

$$\mathbb{P}\left(\sigma_{\tilde{n}} - \sigma_n \geq \frac{n^{1-\alpha} - \tilde{n}^{1-\alpha} - n^{1-\alpha-\beta}}{\bar{\rho}\bar{\kappa}(\alpha-1)} \mid \mathcal{F}_{\sigma_n}\right) = 1 - o_n^\infty(n), \quad (4.21)$$

at a deterministic rate independent of the particular choice of \tilde{n} . If $\alpha = 1$, we instead have

$$\mathbb{P}\left(\sigma_{\tilde{n}} - \sigma_n \geq (\bar{\rho}\bar{\kappa})^{-1} \log\left(\frac{\tilde{n}}{n}\right) + n^{-\beta} \mid \mathcal{F}_{\sigma_n}\right) = 1 - o_n^\infty(n), \quad (4.22)$$

at a deterministic rate independent of the particular choice of \tilde{n} .

Proof. We treat the case where $\alpha > 1$; the case where $\alpha = 1$ is treated similarly. Let U_n be the set of $u \in \mathbb{Z}^d \setminus B_n^{|\cdot|}(0)$ such that u is incident to a vertex in $B_n^{|\cdot|}(0) \cap \mathbb{Z}^d$. For $u \in U_n$, let \mathfrak{A}_u be the subgraph of \mathbb{Z}^d whose vertex set is $(\mathbb{Z}^d \cap B_n^{|\cdot|}(0)) \cup \{u\}$ and whose edge set is the set of all edges in $\mathcal{E}(\mathbb{Z}^d)$ which join vertices in its vertex set. Almost surely, the set A_{σ_n} contains exactly one element of U_n and no other elements of $\mathbb{Z}^d \setminus B_n^{|\cdot|}(0)$. By Lemma 2.4, the conditional law of $\sigma_{\tilde{n}} - \sigma_n$ given \mathcal{F}_{σ_n} a.s. stochastically dominates the conditional law of $\sigma_{\tilde{n}} - \sigma_n$ given $\{A_{\sigma_n} = \mathfrak{A}_u\}$ for some $u \in U_n$. Hence to prove (4.21), it suffices to show that

$$\mathbb{P}\left(\sigma_{\tilde{n}} - \sigma_n \geq \frac{n^{1-\alpha} - \tilde{n}^{1-\alpha} - n^{(1-\beta)(1-\alpha)}}{\rho\kappa(\alpha-1)} \mid A_{\sigma_n} = \mathfrak{A}_u\right) = 1 - o_n^\infty(n), \quad (4.23)$$

uniformly over all choices of $u \in U_n$.

To prove (4.23), let $v_{\tilde{n}}$ be the (a.s. unique) element of $\mathcal{V}(A_{\sigma_{\tilde{n}}}) \setminus B_{\tilde{n}}^{|\cdot|}(0)$. By (4.19),

$$D(v_{\tilde{n}}, \mathfrak{A}_u) \geq \frac{n^{1-\alpha} - \tilde{n}^{1-\alpha}}{\rho\kappa(\alpha-1)} - O_n(n^{-\alpha}), \quad \forall u \in U_n.$$

Furthermore, on the event $\{A_{\sigma_n} = \mathfrak{A}_u\}$ for $u \in U_n$, the FPP geodesic $\eta_{v_{\tilde{n}}}$ from 0 to $v_{\tilde{n}}$ satisfies

$$\eta_{v_{\tilde{n}}} \setminus \mathcal{E}(A_{\sigma_n}) \subset B_{(R+1)n}^D(0) \setminus B_n^{|\cdot|}(0).$$

Therefore, Lemma 3.5 (applied with n in place of m and ξ slightly larger than 1) implies that for each $\beta \in (0, 1/3)$, the following is true. For each $u \in U_n$, it holds except on an event of conditional probability $1 - o_n^\infty(n)$ given $\{A_{\sigma_n} = \mathfrak{A}_u\}$ (at a rate independent from u) that

$$\begin{aligned} \sigma_{\tilde{n}} - \sigma_n &= T(0, v_{\tilde{n}}) - \sigma_n \geq (1 - n^{-\beta}) \left(\frac{\hat{n}^{1-\alpha} - n^{1-\alpha}}{\rho\kappa(\alpha-1)} - O_n(n^{-\alpha}) \right) - O_n(n^{1-\beta-\alpha}) \\ &\geq \frac{\hat{n}^{1-\alpha} - n^{1-\alpha} - O_n(n^{1-\alpha-\beta})}{\rho\kappa(\alpha-1)}. \end{aligned} \quad (4.24)$$

We obtain (4.23) by slightly increasing β , which completes the proof of (4.21). \square

The following lemma tells us that vertices of \mathbb{Z}^d sufficiently close to A_{σ_n} are likely to be absorbed by the FPP clusters before time $\sigma_{\tilde{n}}$, for $\tilde{n} \geq n$ with $\tilde{n} \asymp n$.

Lemma 4.6. *Fix $R \geq 2$ and $\beta \in (0, 1/3)$. Suppose $n \in \mathbb{N}$ and $\tilde{n} \in [(1 + R^{-1})n, Rn]_{\mathbb{Z}}$. For $\alpha > 1$, let*

$$V_{n, \tilde{n}} := \left\{ v \in \mathbb{Z}^d \cap \left(B_n^{|\cdot|}(0) \setminus B_{n/2}^{|\cdot|}(0) \right) : D(v, A_{\sigma_n}) \leq \frac{n^{1-\alpha} - \tilde{n}^{1-\alpha} - n^{1-\alpha-\beta}}{\rho\kappa(1-\alpha)} \right\}. \quad (4.25)$$

For $\alpha = 1$, instead let

$$V_{n, \tilde{n}} := \left\{ v \in \mathbb{Z}^d \cap \left(B_n^{|\cdot|}(0) \setminus B_{R^{-1}n}^{|\cdot|}(0) \right) : D(v, A_{\sigma_n}) \leq \bar{\rho}^{-1} \kappa^{-1} \log \left(\frac{\tilde{n}}{n} \right) - n^{-\beta} \right\}. \quad (4.26)$$

Then

$$\mathbb{P}(V_{n, \tilde{n}} \subset \mathcal{V}(A_{\sigma_{\tilde{n}}}) \mid \mathcal{F}_{\sigma_n}) = 1 - o_n^\infty(n), \quad (4.27)$$

at a deterministic rate independent of the particular choice of \tilde{n} .

Proof. We treat the case where $\alpha > 1$; the case where $\alpha = 1$ is treated similarly. Let $v \in V_{n, \tilde{n}}$ be chosen in some \mathcal{F}_{σ_n} -measurable manner. For each $\epsilon > 0$, we can find a piecewise linear path γ connecting v to some vertex in $\mathcal{V}(A_{\sigma_n})$ with

$$\text{len}^D(\gamma) \leq \frac{n^{1-\alpha} - \tilde{n}^{1-\alpha} - n^{1-\alpha-\beta} + O_n(n^{-\alpha})}{\rho\kappa(1-\alpha)} + \epsilon. \quad (4.28)$$

Observe that for small enough ϵ and large enough n , γ cannot exit $B_{2\tilde{n}}^{|\cdot|}(0)$. Indeed, if it did, then by (4.19) we would have

$$\text{len}^D(\gamma) \geq \frac{n^{1-\alpha} - (2\tilde{n})^{1-\alpha}}{\rho\kappa(1-\alpha)},$$

which is larger than the right side of (4.28) for large enough n and small enough ϵ . On the other hand, if $\delta \in (0, R^{-1})$ and γ enters $B_{\delta n}^{|\cdot|}(0)$, then

$$\text{len}^D(\gamma) \geq \frac{(\delta n)^{1-\alpha} - (R^{-1}n)^{1-\alpha}}{\bar{\rho}\bar{\kappa}(1-\alpha)},$$

which is larger than the right side of (4.28) for large enough n and small enough ϵ provided δ is chosen sufficiently small, depending only R .

It therefore follows from Lemma 3.4 that for each $\beta' \in (\beta, 1/3)$, it holds except on an event of conditional probability $1 - o_n^\infty(n)$ given \mathcal{F}_{σ_n} that

$$\begin{aligned} T(0, v) - \sigma_n &\leq \left(1 + n^{-\beta'}\right) \frac{n^{1-\alpha} - \tilde{n}^{1-\alpha} - n^{1-\alpha-\beta} + O_n(n^{-\alpha})}{\bar{\rho}\bar{\kappa}(1-\alpha)} + n^{1-\beta'-\alpha} \\ &\leq \frac{n^{1-\alpha} - \tilde{n}^{1-\alpha} - n^{1-\alpha-\beta}}{\bar{\rho}\bar{\kappa}(1-\alpha)} + O_n(n^{1-\alpha-\beta'}). \end{aligned} \quad (4.29)$$

For large enough n (how large is deterministic and independent from v), the right side of this last inequality is smaller than our lower bound for $\sigma_{\tilde{n}} - \sigma_n$ from Lemma 4.5. Hence $v \in \mathcal{V}(A_{\sigma_{\tilde{n}}})$ except on an event of conditional probability $1 - o_n^\infty(n)$ given \mathcal{F}_{σ_n} . We conclude by means of the union bound. \square

Proof of Theorem 1.9. The statement of the theorem is immediate from Theorem 1.7 in the case where $\alpha < 1$, so we can assume without loss of generality that $\alpha \in [1, 1 + (\bar{\rho}\bar{\kappa}\lambda)^{-1}]$. Fix $R \geq 2$ and for $n, \tilde{n} \in \mathbb{N}$, let $V_{n, \tilde{n}}$ be as in Lemma 4.6. Also let v_n be the (a.s. unique) point of $A_{\sigma_n} \setminus B_n^{|\cdot|}(0)$. By (4.20), if $v \in \mathbb{Z}^d$ and we let $n_v = \lceil |v| \rceil$, then

$$D(v_{n_v}, v) \leq \lambda n_v^{1-\alpha} + O_{n_v}(n_v^{-\alpha}).$$

By our assumption on α , we have $\lambda < \frac{1}{\bar{\rho}\bar{\kappa}(\alpha-1)}$. Therefore, we can find $n_* \in \mathbb{N}$ and $R \geq 2$ (depending only on μ, ν , and α) such that if $n_v \geq n_*$ and $\tilde{n}_v := \lfloor R n_v \rfloor$, then $v \in V_{n_v, \tilde{n}_v}$. By Lemma 4.6, we obtain (1.17). The second assertion follows from the first assertion and the Borel-Cantelli lemma. \square

5 Proof of cone containment result

In this section we will prove Theorem 1.8. Throughout this section, we always assume $\alpha > 1$.

In Section 5.1, we will define the class of α -weight functions f which we will consider (which in particular includes the α -th powers of a certain family of norms) and state a more quantitative version of Theorem 1.8 (namely Theorem 5.2). We will give an outline of the content of the rest of this section just after the statement of Theorem 5.2.

We remark that the main difficulty in the proof of Theorem 1.8 is geometric, rather than probabilistic. In particular, we do not have good estimates for the deterministic metric D of (1.10) unless f takes a rather specific form. The primary reason for this problem is that very little is known about the Eden model limit shape \mathbb{A} and the corresponding metric μ .

5.1 Cylindrical convex sets and admissible weight functions

In this section we will define the class of α -weight functions for which we will prove our cone containment result. We start by defining the set of norms whose α -th powers are contained in this class. See Figure 4 for an illustration of the unit ball of such a norm when $d = 2$.

Recall the definition of the constant $\bar{\rho}$ and the set \mathbb{X} of maximal μ -unit vectors from Definition 1.4. For $\mathbf{x} \in \mathbb{X}$, let $P_{\mathbf{x}}$ be the $d-1$ -hyperplane containing \mathbf{x} which is perpendicular to the line through 0 and \mathbf{x} . Note that $P_{\mathbf{x}}$ intersects $B_{\bar{\rho}}^{|\cdot|}(0)$ only at \mathbf{x} and $\mathbb{A} \subset B_{\bar{\rho}}^{|\cdot|}(0)$, so $P_{\mathbf{x}} \cap \mathbb{A} = \{\mathbf{x}\}$. Let $P_{\mathbf{x}}^0 := P_{\mathbf{x}} - \mathbf{x}$ be the hyperplane through 0 perpendicular to the line through 0 and \mathbf{x} .

Fix $\mathbf{x} \in \mathbb{X}$ and let Q be a compact convex subset of $P_{\mathbf{x}}^0$ which contains $B_{\bar{\rho}}^{|\cdot|}(0) \cap P_{\mathbf{x}}^0$ and is symmetric about the origin. For $s > 1$, let

$$\mathcal{Q}_s := \{sz + t\mathbf{x} : z \in Q, t \in [-1, 1]\} \quad (5.1)$$

be the cylinder of Euclidean height $2\bar{\rho}$ over sQ . The set \mathcal{Q}_s is compact, convex, and symmetric about the origin so

$$\nu_s(z) := \inf \{r > 0 : z \in r\mathcal{Q}_s\} \quad \forall z \in \mathbb{R}^d \quad (5.2)$$

defines a norm on \mathbb{R}^d whose unit ball is \mathcal{Q}_s .

We note that the set $\partial\mathcal{Q}_s = \partial B_1^{\nu_s}(0)$ possesses two distinguished flat faces, namely $\partial\mathcal{Q}_s \cap P_{\mathbf{x}}$ and $\partial\mathcal{Q}_s \cap P_{-\mathbf{x}}$, which are reflections of each other through the origin. Due to our choice of \mathbf{x} and since $s > 1$, the set $\mathbb{A} \cap \partial\mathcal{Q}_s$ consists of two points, one of which belongs to each of these two distinguished flat faces.

Let f be an α -weight function and for $s > 1$ let $f_s := f|_{\partial\mathcal{Q}_s}$. Then

$$f(z) = \nu_s(z)^\alpha f_s\left(\frac{z}{\nu_s(z)}\right), \quad \forall z \in \mathbb{R}^d \quad (5.3)$$

so we can represent f by means of the parameters f_s and α , rather than f_0 and α from (1.1). Note that f_s is Lipschitz continuous if and only if f_0 is Lipschitz continuous.

Definition 5.1. For f_s as above, write

$$\bar{\kappa}_s := \sup_{z \in \partial\mathcal{Q}_s} f_s(z) \quad \text{and} \quad \underline{\kappa}_s := \inf_{z \in \partial\mathcal{Q}_s} f_s(z). \quad (5.4)$$

We say that $f_s : \partial\mathcal{Q}_s \rightarrow (0, \infty)$ is *admissible* if f_s is Lipschitz continuous and $f_s \equiv \bar{\kappa}_s$ on $\partial\mathcal{Q}_s \cap (P_{\mathbf{x}} \cup P_{-\mathbf{x}})$.

Henceforth fix $s > 1$ and an admissible function f_s and let f as in (5.3). Note that constant functions are admissible in the sense of Definition 5.1, so we can take f to be the α -th power of the norm ν_s . Let $\{A_t\}_{t \geq 0}$ be the f -weighted FPP clusters, as in Section 1.3. The main goal of this section is to prove the following theorem, which immediately implies Theorem 1.8.

Theorem 5.2. Suppose $\alpha > 1$, $s > 1$, Q , \mathcal{Q}_s , and f_s are as above. Let

$$\mathcal{K} := \bigcup_{r>0} r(\partial\mathcal{Q}_s \cap P_{\mathbf{x}})$$

and note that \mathcal{K} is contained in a Euclidean cone of opening angle $< \pi$. If

$$s > 2^{\frac{\alpha}{\alpha-1}} - 1, \quad (5.5)$$

then

$$\mathbb{P}(\#(\mathcal{V}(A_{\tau_\infty}) \setminus \mathcal{K}) < \infty) > 0 \quad \text{and} \quad \mathbb{P}(\#(\mathcal{V}(A_{\tau_\infty}) \setminus (-\mathcal{K})) < \infty) > 0. \quad (5.6)$$

If, in addition,

$$s > 1 + \frac{\bar{\kappa}_s \alpha^\alpha}{\underline{\kappa}_s (\alpha - 1)^{\alpha-1}} \quad (5.7)$$

then a.s. either

$$\#(\mathcal{V}(A_{\tau_\infty}) \setminus \mathcal{K}) < \infty \quad \text{or} \quad \#(\mathcal{V}(A_{\tau_\infty}) \setminus (-\mathcal{K})) < \infty. \quad (5.8)$$

In the rest of this section we will prove Theorem 5.2. We now give an outline of the proof.

In Section 5.2, we will prove explicit bounds for various distances with respect to the metric D of (1.10) defined with the above choice of α -weight function f , via elementary geometric arguments. Due to the particular form of f , we will be able to obtain such estimates even without knowing the form of the standard FPP limit shape \mathbb{A} . In particular, we will obtain an explicit formula for the distance between the union of the distinguished flat faces $\partial\mathcal{Q}_s \cap (P_{\mathbf{x}} \cup P_{-\mathbf{x}})$ and its scaling $q(\partial\mathcal{Q}_s \cap (P_{\mathbf{x}} \cup P_{-\mathbf{x}}))$ for $q > 1$; and show that the minimum distance is attained along segments perpendicular to \mathbf{x} (Lemma 5.4). We also prove upper and lower bounds for distances between arbitrary given points of \mathcal{Q}_s and $q\partial\mathcal{Q}_s$ (Lemma 5.6).

In Section 5.3, we will use the estimates of Sections 3.3 and 5.2 to prove estimates for the f -weighted FPP clusters A_t . The most important estimate of Section 5.3 is Lemma 5.9, which will tell us, roughly speaking, that the following holds. If for some large $n_0 \in \mathbb{N}$, the exit position of the clusters $\{A_t\}_{t \geq 0}$ from $B_{n_0}^{\nu_s}(0)$ lies in $n_0(\partial\mathcal{Q}_s \cap (P_{\mathbf{x}} \cup P_{-\mathbf{x}}))$ (up to rounding error), then it is likely that the following is true for each $n \in \mathbb{N}$ a little bit bigger than n_0 .

- The exit position of $\{A_t\}_{t \geq 0}$ from $B_n^{\nu_s}(0)$ lies in $n_0(\partial\mathcal{Q}_s \cap (P_{\mathbf{x}} \cup P_{-\mathbf{x}}))$ (up to rounding error).
- A_{τ_∞} does not contain any vertices of \mathbb{Z}^d which lie at D -distance greater than a constant times $n^{1-\alpha}$ from $B_n^{\nu_s}(0)$.

We will also show that if (5.7) holds, then for large enough $n_0 \in \mathbb{N}$ it is likely that the clusters first exit $B_{n_0}^{\nu_s}(0)$ at a point near $n_0(\partial\mathcal{Q}_s \cap (P_{\mathbf{x}} \cup P_{-\mathbf{x}}))$ (see Lemma 5.10), so the above two conditions are likely to hold for all large enough n . The proof of these estimates is inductive in nature, and relies crucially on the precise estimates for D in Section 5.2 to control the exit position of the clusters from $B_n^{\nu_s}(0)$.

In Section 5.4, we will use the estimates of Section 5.3 to conclude that if s is chosen appropriately, then a.s. all but finitely many vertices of A_{τ_∞} are contained in $\mathcal{K} \cup (-\mathcal{K})$, in the notation of Theorem 5.2. We will then use Proposition 2.7 to rule out the possibility that there are infinitely many vertices of A_{τ_∞} contained in each of \mathcal{K} and $-\mathcal{K}$.

5.2 Geometric estimates for cylindrical sets

In this subsection, we will prove some deterministic geometric properties of the metric D associated with a general admissible function $f_s : \partial\mathcal{Q}_s \rightarrow (0, \infty)$ (Definition 5.1). Throughout, we fix $\alpha > 1$, $s > 1$, and an admissible function f_s and use the notation introduced in Section 5.1 and we let D be as in (1.10) with $f = f_s$. Our main focus is on estimating distances in the metric D so that we can eventually apply the results of Section 3.3 to prove Theorem 1.8. See Figure 7 for an illustration of the key idea of this subsection.

We start by collecting some basic properties of the set \mathcal{Q}_s and its associated norm ν_s . For the statement, we recall the definition of the constant $\bar{\rho}$ from Definition 1.4.

Lemma 5.3. *Suppose \mathcal{Q}_s is as in (5.1) and ν_s is as in (5.2). Then the following holds.*

1. $B_1^\mu(0) \subset B_{\bar{\rho}}^{|\cdot|}(0) \subset \mathcal{Q}_s$, so $\mu(w - z) \geq \bar{\rho}^{-1}|w - z| \geq \nu_s(w - z)$ for each $z, w \in \mathbb{R}^d$.
2. For each $q > 1$,
$$\text{dist}^{|\cdot|}(\mathcal{Q}_s, q(\partial\mathcal{Q}_s \setminus (P_{\mathbf{x}} \cup P_{-\mathbf{x}}))) \geq \bar{\rho}s(q - 1). \quad (5.9)$$
3. For each $q > 1$, we have $\text{dist}^{|\cdot|}(\partial\mathcal{Q}_s, q\partial\mathcal{Q}_s) = \bar{\rho}(q - 1)$.
4. For each $q > 1$, $z \in \partial\mathcal{Q}_s$, and $w \in q\partial\mathcal{Q}_s$ with $|w - z| = \bar{\rho}(q - 1)$, we have $z \in P_{\mathbf{x}} \cup P_{-\mathbf{x}}$ and $w = z \pm (q - 1)\mathbf{x}$.

Proof. We first check assertion 1. Suppose $w \in B_{\bar{\rho}}^{|\cdot|}(0)$ and let w^\perp be its projection onto the plane $P_{\mathbf{x}}^0$ through 0 perpendicular to \mathbf{x} . Then $|w^\perp| \leq |w|$ so $w^\perp \in B_{\bar{\rho}}^{|\cdot|}(0) \cap P_{\mathbf{x}}^0 \subset Q$. Furthermore, $w - w^\perp = t\mathbf{x}$ where $t \in \mathbb{R}$ with $|t| = \bar{\rho}^{-1}|w - w^\perp| \leq \bar{\rho}$. Since $B_{\bar{\rho}}^{|\cdot|}(0) \cap P_{\mathbf{x}}^0 \subset Q$, it therefore follows from (5.1) that $w = w^\perp + t\mathbf{x} \in \mathcal{Q}_s$. By definition of $\bar{\rho}$ we have $B_1^\mu(0) = \mathbb{A} \subset B_{\bar{\rho}}^{|\cdot|}(0)$, and the statement about norms is immediate from (5.2).

We next observe that for $q > 1$, each point of $q\partial\mathcal{Q}_s$ lies at ν_s -distance $q - 1$ from $\partial\mathcal{Q}_s$. Since $B_{\bar{\rho}}^{|\cdot|}(0) \cap P_{\mathbf{x}}^0 \subset Q$, each such point lies at Euclidean distance at least $\bar{\rho}(q - 1)$ from Q . From (5.1), we now obtain assertion 2. Since $|\mathbf{x}| = \rho$, we have $\text{dist}^{|\cdot|}(P_{\mathbf{x}} \cup P_{-\mathbf{x}}, q(P_{\mathbf{x}} \cup P_{-\mathbf{x}})) = \bar{\rho}(q - 1)$ and $|q\mathbf{x} - \mathbf{x}| = \bar{\rho}(q - 1)$. By combining this with assertion 2, we obtain assertion 3.

Now suppose $q > 1$, $z \in \partial\mathcal{Q}_s$, and $w \in q\partial\mathcal{Q}_s$ with $|w - z| = \bar{\rho}(q - 1)$. By assertions 2 and 3 we have $w \in q(P_{\mathbf{x}} \cup P_{-\mathbf{x}})$. By symmetry we can assume without loss of generality that $w \in qP_{\mathbf{x}}$. Any path from w to $\mathcal{Q}_s \setminus P_{\mathbf{x}}$ must pass through $P_{\mathbf{x}}$, so must have Euclidean length $> \bar{\rho}(q - 1)$. Therefore $z \in P_{\mathbf{x}}$. The unique closest point to z in $qP_{\mathbf{x}}$ is $z + (q - 1)\mathbf{x}$, so we must in fact have $w = z + (q - 1)\mathbf{x}$. \square

Our next lemma generalizes some of the statements of Lemma 5.3 to the metric D .

Lemma 5.4. *For each $q > 1$ and each $z \in P_{\mathbf{x}} \cup P_{-\mathbf{x}}$,*

$$D(\partial\mathcal{Q}_s, q\partial\mathcal{Q}_s) = D(z, z + (q - 1)\mathbf{x}) = \frac{1 - q^{1-\alpha}}{\bar{\kappa}_s(\alpha - 1)}. \quad (5.10)$$

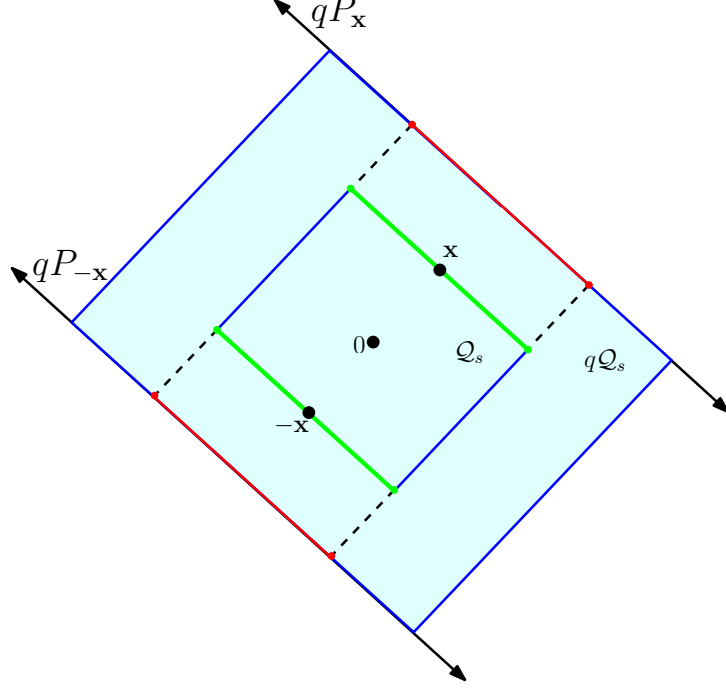


Figure 7: An illustration of the convex sets \mathcal{Q}_s and $q\mathcal{Q}_s$ for $q > 1$. Lemma 5.4 shows that every path of minimal D -length from $\partial\mathcal{Q}_s$ to $q\partial\mathcal{Q}_s$ lies between one of the two pairs of dotted lines shown in the figure. In particular, every point in $\partial\mathcal{Q}_s$ which lies at minimal D -distance from $q\partial\mathcal{Q}_s$ belongs to one of the green lines and every point in $q\partial\mathcal{Q}_s$ which lies at minimal D -distance from $\partial\mathcal{Q}_s$ belongs to one of the red lines. Hence, if we re-scale by q^{-1} so that $q\partial\mathcal{Q}_s$ is mapped to $\partial\mathcal{Q}_s$, then the red lines are mapped to proper subsets of the green lines. This means that for each small $\delta > 0$, each point of $q\partial\mathcal{Q}_s$ which lies within D -distance at most $D(\partial\mathcal{Q}_s, q\partial\mathcal{Q}_s) + \delta$ from $\partial\mathcal{Q}_s$ lies at D -distance exactly $q^{1-\alpha}D(q\partial\mathcal{Q}_s, q^2\partial\mathcal{Q}_s)$ from $q^2\partial\mathcal{Q}_s$. We also have the following facts, which come from Lemma 5.6. If s is chosen so that (5.5) holds, then for large enough q the D -distance from \mathcal{Q}_s to any point in $q(\partial\mathcal{Q}_s \setminus (P_{\mathbf{x}} \cup P_{-\mathbf{x}}))$ is greater than the D -distance from \mathcal{Q}_s to ∞ . Furthermore, if s is chosen so that (5.7) holds, then for an appropriate choice of q the D -distance from any point of $\partial\mathcal{Q}_s$ to $q(\partial\mathcal{Q}_s \setminus (P_{\mathbf{x}} \cup P_{-\mathbf{x}}))$ is greater than its D -distance to $q(\partial\mathcal{Q}_s \cap (P_{\mathbf{x}} \cup P_{-\mathbf{x}}))$. The above observations together with the estimates of Section 3.3 and an induction argument will be used to prove Theorem 5.2.

Furthermore, if $q > 1$, $z \in \partial\mathcal{Q}_s$, and $w \in q\partial\mathcal{Q}_s$ with $D(z, w) = D(\partial\mathcal{Q}_s, q\partial\mathcal{Q}_s)$ then $z \in P_{\mathbf{x}} \cup P_{-\mathbf{x}}$ and $w = z \pm (q-1)\mathbf{x}$.

Proof. First suppose $q > 1$ and $z \in P_{\mathbf{x}} \cap \partial\mathcal{Q}_s$. For $t \in [0, q-1]$ let $\ell(t) := z + t\mathbf{x}$. Then ℓ is a linear path parametrized by μ -length and $\nu_s(\ell(t)) = 1+t$ for each $t \in [0, q-1]$. Furthermore, for each such t we have $(1+t)^{-1}\ell(t) \in P_{\mathbf{x}} \cap \partial\mathcal{Q}_s$.

Therefore, $f(\ell(t)) = (1+t)^\alpha \bar{\kappa}_s$ for each $t \in [0, q-1]$ and

$$D(\partial\mathcal{Q}_s, q\partial\mathcal{Q}_s) \leq D(z, z + (q-1)\mathbf{x}) \leq \text{len}^D(\ell) = \int_0^{q-1} (1+t)^{-\alpha} \bar{\kappa}_s^{-1} dt = \frac{1 - q^{1-\alpha}}{\bar{\kappa}_s(\alpha-1)}. \quad (5.11)$$

Now suppose $q > 1$, $z \in \partial\mathcal{Q}_s$, and $w \in q\partial\mathcal{Q}_s$. Let γ be a piecewise linear path from z to w , parametrized by μ -length. Let $T := \text{len}^\mu(\gamma)$. By assertion 1 of Lemma 5.3, for each $t \in [0, T]$ we have $\text{len}^{\nu_s}(\gamma([0, t])) \leq t$, so $\nu_s(\gamma(t)) \leq 1+t$. Therefore $f(\gamma(t)) \leq (1+t)^\alpha \bar{\kappa}_s$ for each $t \in [0, q-1]$, so

$$\text{len}^D(\gamma) \geq \int_0^T (1+t)^{-\alpha} \bar{\kappa}_s^{-1} dt \geq \frac{1 - (1+T)^{1-\alpha}}{\bar{\kappa}_s(\alpha-1)}. \quad (5.12)$$

By assertion 3 of Lemma 5.3, $\text{len}^{|\cdot|}(\gamma) \geq \bar{\rho}(q-1)$, so $T \geq q-1$. Furthermore, we have strict inequality unless $z \in P_{\mathbf{x}} \cup P_{-\mathbf{x}}$ and $w = z \pm (q-1)\mathbf{x}$. By combining (5.11) and (5.12), we obtain the statement of the lemma. \square

To complement Lemma 5.4, we also have a lower bound for the distance from \mathcal{Q}_s to points of $q\partial\mathcal{Q}_s$ which are not translates of elements of $\partial\mathcal{Q}_s \cap (P_{\mathbf{x}} \cup P_{-\mathbf{x}})$ in a direction perpendicular to \mathbf{x} .

Lemma 5.5. *For $q > 1$, let*

$$K_q := \{z + (q-1)\mathbf{x} : z \in P_{\mathbf{x}} \cap \partial\mathcal{Q}_s\} \cup \{z - (q-1)\mathbf{x} : z \in P_{-\mathbf{x}} \cap \partial\mathcal{Q}_s\}.$$

For each $q_2 > q_1 > 1$ and $\epsilon > 0$, there exists $\delta = \delta(f, q_1, q_2, \epsilon) > 0$ such that for each $q \in [q_1, q_2]$ and each $w \in q\partial\mathcal{Q}$ with $\text{dist}^{|\cdot|}(w, K_{1,q}(\mathbf{x})) \geq \epsilon$,

$$D(w, \partial\mathcal{Q}_s) \geq \frac{1 - q^{1-\alpha}}{\bar{\kappa}_s(\alpha - 1)} + \delta.$$

Proof. Let

$$K := \bigcup_{q=q_1}^{q_2} q\partial\mathcal{Q}_s \quad \text{and} \quad K' := \bigcup_{q=q_1}^{q_2} K_q.$$

Note that K is compact and $K' \subset K$. Define $\psi : K \rightarrow (0, \infty)$ by

$$\psi(w) = D(w, \partial\mathcal{Q}_s) - \frac{1 - \nu_s(w)^{1-\alpha}}{\bar{\kappa}_s(\alpha - 1)}.$$

By Lemma 3.3 and since the norms $|\cdot|$ and ν_s are comparable, we infer that ψ is Euclidean-continuous. By Lemma 5.4, $\psi(w) > 0$ for each $w \in K \setminus K'$. By compactness, there is a $\delta > 0$ such that $\psi(w) \geq \delta$ for each $w \in K$ with $\text{dist}^{|\cdot|}(w, K') \geq \epsilon$. If ϵ is chosen sufficiently small (depending only on q_1, q_2 , and f), then whenever $w \in q\partial\mathcal{Q}_s$ for $q \in [q_1, q_2]$ and $\text{dist}^{|\cdot|}(w, K_q) \geq \epsilon$, we also have $\text{dist}^{|\cdot|}(w, K') \geq \epsilon$. The statement of the lemma follows. \square

Our next lemma is the source of the conditions (5.5) and (5.7) in Theorem 5.2. To state one of the estimates in the lemma, we will need the following notation. For $r > 0$, and $z, w \in B_r^{\nu_s}(0)$, we define a modified version of the metric D by

$$D_r(z, w) := \inf \left\{ \text{len}^D(\gamma) : \gamma \text{ is a piecewise linear path from } z \text{ to } w \text{ contained in } B_r^{\nu_s}(0) \right\}. \quad (5.13)$$

We recall that similar modifications of D appear in Lemma 3.4 and Remark 3.8.

Lemma 5.6. *Let $q > 1$.*

1. For each $z \in \partial\mathcal{Q}_s$, there exists $c \in \mathbb{R}$ such that $z + c\mathbf{x} \in q(P_{\mathbf{x}} \cup P_{-\mathbf{x}})$ and

$$D(z, z + c\mathbf{x}) \leq \frac{1 - q^{1-\alpha}}{\bar{\kappa}_s(\alpha - 1)} + \frac{1}{\underline{\kappa}_s}. \quad (5.14)$$

2. For each $\zeta > 0$, each $z \in \mathcal{Q}_s$, and each $w \in q(\partial\mathcal{Q}_s \setminus (P_{\mathbf{x}} \cup P_{-\mathbf{x}}))$, we have (in the notation (5.13))

$$D_{q+\zeta}(z, w) \geq \frac{1 - (q + \zeta)^{1-\alpha}}{\bar{\kappa}_s(\alpha - 1)} + \frac{(s-1)(q-1) - \zeta}{\bar{\kappa}_s(q + \zeta)^\alpha}. \quad (5.15)$$

3. For each $z \in \mathcal{Q}_s$ and each $w \in q(\partial\mathcal{Q}_s \setminus (P_{\mathbf{x}} \cup P_{-\mathbf{x}}))$,

$$D(z, w) \geq \frac{1 + q^{1-\alpha} - 2^\alpha(q + 1 + s(q-1))^{1-\alpha}}{\bar{\kappa}_s(\alpha - 1)}. \quad (5.16)$$

Proof. First consider the setting of assertion 1. Assume without loss of generality that z is closer to $P_{\mathbf{x}}$ than $P_{-\mathbf{x}}$ in the Euclidean distance and let $\tilde{c} \geq 0$ be chosen so that $z + \tilde{c}\mathbf{x} \in P_{\mathbf{x}}$. By the definition (5.1) of \mathcal{Q}_s , we have $\tilde{c} \leq 1$. Furthermore, for each $t \in [0, \tilde{c}]$ we have $z + t\mathbf{x} \in \partial\mathcal{Q}_s$, so $f(z + t\mathbf{x}) \geq \underline{\kappa}_s$. By integrating along the path $t \mapsto z + t\mathbf{x}$ (which is parametrized by μ -length) we obtain $D(z, z + \tilde{c}\mathbf{x}) \leq \underline{\kappa}_s^{-1}$. By Lemma 5.4,

$$D(z + \tilde{c}\mathbf{x}, z + c\mathbf{x}) \leq \frac{1 - q^{1-\alpha}}{\bar{\kappa}_s(\alpha - 1)}$$

for $c = \tilde{c} + q - 1$. The estimate (5.14) follows.

Now suppose we are in the setting of assertion 2. Let γ be a piecewise linear path from z to w which is contained in $B_{q+\zeta}^{\nu_s}(0)$ and write $T = \text{len}^\mu(\gamma)$. By assertion 2 of Lemma 5.3, $T \geq s(q - 1)$. Furthermore, for each $t \in [0, T]$ we have $\nu_s(\gamma(t)) \leq (1 + t) \wedge (q + \zeta)$, so

$$f(\gamma(t)) \leq \begin{cases} \bar{\kappa}_s(1 + t)^\alpha, & t \in [0, q - 1 + \zeta] \\ \bar{\kappa}_s(q + \zeta)^\alpha, & t \in [q - 1 + \zeta, T]. \end{cases}$$

Integrating, we get that $\text{len}^D(\gamma)$ is at least the right side of (5.15).

Finally, we consider the setting of assertion 3. Let $\gamma : [0, T] \rightarrow \mathbb{R}^d$ be a piecewise linear path from z to w parametrized by μ -length. As above, $T \geq s(q - 1)$ and $\nu_s(\gamma(t)) \leq 1 + t$ for each $t \in [0, T]$. We can no longer say that $\nu_s(\gamma(t)) \leq q$, since γ may not stay in $q\mathcal{Q}_s$. However, $\gamma(T) = q$ so $\nu_s(\gamma(t)) \leq q + T - t$ for each $t \in [0, T]$. Set $r = (q - 1 + T)/2$. Then

$$\begin{aligned} \text{len}^D(\gamma) &\geq \frac{1}{\bar{\kappa}_s} \int_0^r (1 + t)^{-\alpha} dt + \frac{1}{\bar{\kappa}_s} \int_r^T (q + T - t)^{-\alpha} dt \\ &= \frac{1 + q^{1-\alpha} - 2^\alpha(q + T + 1)^{1-\alpha}}{\bar{\kappa}_s(\alpha - 1)} \\ &\geq \frac{1 + q^{1-\alpha} - 2^\alpha(q + 1 + s(q - 1))^{1-\alpha}}{\bar{\kappa}_s(\alpha - 1)}. \end{aligned} \quad \square$$

5.3 Probabilistic estimates

Throughout this subsection, we fix $s > 1$ and an admissible function $f_s : \partial\mathcal{Q}_s \rightarrow (0, \infty)$. We continue to use the notation of Section 5.1. Let $\{A_t\}_{t \geq 0}$ be the f -weighted FPP process with f as in (5.3) and let $\{\mathcal{F}_t\}_{t \geq 0}$ be the associated filtration as in Definition 1.3. In what follows we will combine the estimates of Sections 3.3 and 5.2 to prove some lemmas about the asymptotic behavior of the clusters $\{A_t\}_{t \geq 0}$. These lemmas will be used to prove Theorem 5.2 in the next subsection.

For the results in this subsection, we introduce the following additional notation. For $r > 0$, let

$$\tau_r := \inf \{t \geq 0 : A_t \not\subset B_r^{\nu_s}(0)\}.$$

Also let u_r be the (a.s. unique) vertex in $\mathcal{V}(A_{\tau_r}) \setminus r\mathcal{Q}_s$ and let

$$G_r := \left\{ \frac{u_r}{\nu_s(u_r)} \in \partial\mathcal{Q}_s \cap (P_{\mathbf{x}} \cup P_{-\mathbf{x}}) \right\} \quad (5.17)$$

be the event that the exit position of the clusters A_{τ_r} from $r\mathcal{Q}_s = B_r^{\nu_s}(0)$ is on one of the flat faces of $r\partial\mathcal{Q}_s$ (i.e. the green faces in Figure 7), modulo rounding error.

We start out with some basic exit time estimates which are similar to estimates from Section 4.

Lemma 5.7. *Fix $R \geq 2$ and $\beta \in (0, 1/3)$. Suppose $n \in \mathbb{N}$ and $\tilde{n} \in [(1 + R^{-1})n, Rn]_{\mathbb{Z}}$. Then*

$$\mathbb{P} \left(\tau_{\tilde{n}} - \tau_n \geq \frac{n^{1-\alpha} - \tilde{n}^{1-\alpha} - n^{1-\alpha-\beta}}{\bar{\kappa}_s(\alpha - 1)} \mid \mathcal{F}_{\tau_n} \right) = 1 - o_n^\infty(n), \quad (5.18)$$

$$\mathbb{P} \left(\tau_{\tilde{n}} - \tau_n \leq \frac{n^{1-\alpha} - \tilde{n}^{1-\alpha} + n^{1-\alpha-\beta}}{\bar{\kappa}_s(\alpha - 1)} \mid \mathcal{F}_{\tau_n} \right) \mathbb{1}_{G_n} = (1 - o_n^\infty(n)) \mathbb{1}_{G_n}, \quad (5.19)$$

and

$$\mathbb{P} \left(\tau_{\tilde{n}} - \tau_n \leq \frac{n^{1-\alpha} - \tilde{n}^{1-\alpha} + n^{1-\alpha-\beta}}{\bar{\kappa}_s(\alpha-1)} + \frac{n^{1-\alpha}}{\underline{\kappa}_s} \mid \mathcal{F}_{\tau_n} \right) = 1 - o_n^\infty(n), \quad (5.20)$$

all at a deterministic rate independent of the particular choice of \tilde{n} .

Proof. The estimate (5.18) is deduced from Lemma 3.5 in a similar manner to the analogous estimates in Lemmas 4.2 and 4.5.

To obtain (5.19), suppose G_n occurs. The proof of Lemma 5.4 shows that we can find a vertex $\tilde{u}_n \in \mathbb{Z}^d \setminus \tilde{n}Q$ and a piecewise linear path γ from u_n to \tilde{u}_n contained in $B_{2\tilde{n}}^{\nu_s}(0) \setminus B_{n/2}^{\nu_s}(0)$ which satisfies

$$\text{len}^D(\gamma) \leq \frac{n^{1-\alpha} - \tilde{n}^{1-\alpha}}{\kappa_s(\alpha-1)} + O_n(n^{-\alpha}). \quad (5.21)$$

Indeed, we can take γ to be a small perturbation of a path which traces the line segment $[u_n, u_n + (n - \tilde{n})\mathbf{x}]$. Therefore, the estimate (5.19) follows from Lemma 3.4. The estimate (5.20) is proven in the same manner as (5.19) but with assertion 1 of Lemma 5.6 used in place of Lemma 5.4. \square

Our next lemma tells us that it is very unlikely that vertices are absorbed by the FPP clusters sooner than we would expect after time τ_n .

Lemma 5.8. *Fix $R \geq 2$, $\beta \in (0, 1/3)$, and $\zeta > 0$. Also let $n \in \mathbb{N}$, $\tilde{n} \in [(1 + R^{-1})n, Rn]_{\mathbb{Z}}$, and let $v_* \in \mathbb{Z}^d \cap (B_{\tilde{n}}^{\nu_s}(0) \setminus B_{\zeta n}^{\nu_s}(0))$ be chosen in a \mathcal{F}_{τ_n} -measurable manner. Then with $D_{(1+\zeta)\tilde{n}}$ as in (5.13),*

$$\mathbb{P} \left(T(0, v_*) - \tau_n \geq (1 - n^{-\beta}) D_{(1+\zeta)\tilde{n}} \left(v, \mathcal{V}(A_{\tau_n}) \cup B_{\zeta n}^{\nu_s}(0) \right) - n^{1-\beta-\alpha} \mid \mathcal{F}_{\tau_n} \right) = 1 - o_n^\infty(n)$$

at a deterministic rate depending only on R , β , and ζ .

Proof. Let $v_* \in \mathbb{Z}^d \cap (B_{\tilde{n}}^{\nu_s}(\cdot) \setminus B_{\zeta n}^{\nu_s}(0))$ be chosen in a \mathcal{F}_{τ_n} -measurable manner. Let \mathfrak{A} be a possible realization of A_{τ_n} and let \mathfrak{A}' be the realization of A_{τ_n} obtained by adjoining to \mathfrak{A} each vertex of \mathbb{Z}^d which is contained in $B_{\zeta n}^{\nu_s}(0)$ and each edge of \mathbb{Z}^d which connects two such vertices. By Lemma 2.4, the conditional law of $T(0, v_*) - \tau_n$ given $\{A_{\tau_n} = \mathfrak{A}\}$ stochastically dominates the conditional law of $T(0, v_*) - \tau_n$ given $\{A_{\tau_n} = \mathfrak{A}'\}$. Hence to prove (4.21), it suffices to show that

$$\mathbb{P} \left(T(0, v_*) - \tau_n \geq (1 - n^{-\beta}) D_{(1+\zeta)\tilde{n}} \left(v, \mathcal{V}(A_{\tau_n}) \cup B_{\zeta n}^{\nu_s}(0) \right) - n^{1-\beta-\alpha} \mid A_{\tau_n} = \mathfrak{A}' \right) = 1 - o_n^\infty(n), \quad (5.22)$$

uniformly over all possible realizations \mathfrak{A}' . To see this, suppose $A_{\tau_n} = \mathfrak{A}'$. Then the FPP geodesic η_{v_*} from 0 to v_* satisfies $\eta_{v_*} \setminus \mathcal{E}(\mathfrak{A}') \subset B_{2\tilde{n}}^{\nu_s}(\cdot) \setminus B_{\zeta n}^{\nu_s}(0)$. The estimate (5.22) therefore follows from Lemma 3.5 (c.f. Remark 3.8). \square

The following lemma is the key input in the proof of Theorem 1.8, and will eventually be used to show that the event (5.8) of Theorem 5.2 is very likely to occur provided G_{n_0} occurs for some large $n_0 \in \mathbb{N}$.

Lemma 5.9. *Fix $R \geq 3$ and $\beta \in (0, 1/3)$. For $r \geq 0$, define the event G_r as in (5.17). For each $n_0 \in \mathbb{N}$, on the event G_{n_0} it holds except on an event of conditional probability $o_{n_0}^\infty(n_0)$ (at a deterministic rate) given $\mathcal{F}_{\tau_{n_0}}$ that the following is true.*

1. The event $\bigcap_{n=[(1+R^{-1})n_0]}^\infty G_n$ occurs.
2. For each $n \in \mathbb{N}$ with $n \geq n_0$,

$$\tau_\infty - \tau_n \leq \frac{n^{1-\alpha}}{\bar{\kappa}_s(\alpha-1)} + n^{1-\alpha-\beta}.$$

3. For each $n \in \mathbb{N}$ with $n \geq n_0$ and each $v \in B_{Rn}^{\nu_s}(0) \setminus B_n^{\nu_s}(0)$ with

$$D(v, A_{\tau_n} \cup B_n^{\nu_s}(0)) \geq \frac{n^{1-\alpha}}{\bar{\kappa}_s(\alpha-1)} + 2n^{1-\alpha-\beta},$$

we have $v \notin A_{\tau_\infty}$.

Proof. For $n_0 \in \mathbb{N}$ and $n \in [(1 + R^{-1})n_0, Rn_0]_{\mathbb{Z}}$, write $V_{n_0, n}$ for the set of $v \in B_{n+\bar{\rho}d}^{\nu_s}(0) \setminus B_{(1+R^{-1})n_0}^{\nu_s}(0)$ with $v/\nu_s(v) \notin P_{\mathbf{x}} \cup P_{-\mathbf{x}}$. By Lemma 5.5 and scale invariance, we can find $\delta = \delta(f, R) > 0$ such that for each sufficiently large $n_0 \in \mathbb{N}$, each $n \in [(1 + R^{-1})n_0, Rn_0]_{\mathbb{Z}}$, and each $v \in V_{n_0, n}$,

$$D(v, B_{n_0}^{\nu_s}(0)) \geq \frac{n_0^{1-\alpha} - n^{1-\alpha}}{\bar{\kappa}_s(\alpha - 1)} + \delta n_0^{1-\alpha}.$$

By Lemma 5.8, it holds except on an event of conditional probability $o_{n_0}^\infty(n_0)$ given $\mathcal{F}_{\tau_{n_0}}$ that

$$T(0, v) - \tau_{n_0} \geq \frac{n_0^{1-\alpha} - n^{1-\alpha}}{\bar{\kappa}_s(\alpha - 1)} + \delta n_0^{1-\alpha} - n_0^{1-\alpha-\beta}, \quad \forall v \in V_{n_0, n}. \quad (5.23)$$

By (5.19) of Lemma 5.7, if G_{n_0} occurs then except on an event of conditional probability $o_{n_0}^\infty(n_0)$ given $\mathcal{F}_{\tau_{n_0}}$,

$$\tau_n - \tau_{n_0} \leq \frac{n_0^{1-\alpha} - n^{1-\alpha}}{\bar{\kappa}_s(\alpha - 1)} + n_0^{1-\alpha-\beta},$$

which is smaller than the right side of (5.23) for large enough n_0 . Hence if G_{n_0} occurs, then except on an event of conditional probability $o_{n_0}^\infty(n_0)$ given $\mathcal{F}_{\tau_{n_0}}$, the unique element $u_n \in \mathcal{V}(A_{\tau_n}) \setminus B_n^{\nu_s}(0)$ does not belong to $V_{n_0, n}$, i.e. G_n occurs.

By the union bound, we obtain

$$\mathbb{P} \left(\bigcap_{n=\lceil (1+R^{-1})n_0 \rceil}^{\lfloor Rn_0 \rfloor} G_{n_0} \mid \mathcal{F}_{\tau_{n_0}} \right) \mathbb{1}_{G_{n_0}} = (1 - o_{n_0}^\infty(n_0)) \mathbb{1}_{G_{n_0}}. \quad (5.24)$$

Since $R \geq 3$,

$$[(1 + R^{-1})n_0, \infty) \subset \bigcup_{k=1}^{\infty} [(1 + R^{-1})^k n_0, R^k n_0]_{\mathbb{Z}}.$$

By (5.24) and induction, we infer that condition 1 in the statement of the lemma holds with conditional probability $1 - o_{n_0}^\infty(n_0)$ given $\mathcal{F}_{\tau_{n_0}}$ on G_{n_0} .

The desired estimate for the probability of the event of condition 2 follows from condition 1 together with (5.19) of Lemma 5.7 (the later is applied with $R^k n$ for $k \in \mathbb{N}$ in place of n , and then summed over all $k \in \mathbb{N}$). We slightly shrink β if necessary to allow us to drop an R, α, β -dependent constant in front of $n^{1-\alpha-\beta}$.

To estimate the probability of the event of condition 3, we first apply Lemma 5.8 and the union bound to find that except on an event of conditional probability $o_{n_0}^\infty(n_0)$ given $\mathcal{F}_{\tau_{n_0}}$,

$$T(0, v) - \tau_n \geq \frac{n^{1-\alpha}}{\bar{\kappa}_s(\alpha - 1)} + (2 - o_n(1))n^{1-\alpha-\beta}$$

for each $n \geq n_0$ and each vertex v as in condition 3 (here we use that $D \leq D_{(1+\zeta)\bar{n}}$). By combining this with the condition 2, we find that on G_{n_0} , the conditional probability given $\mathcal{F}_{\tau_{n_0}}$ of the event in condition 3 is at least $1 - o_{n_0}^\infty(n_0)$. \square

In order to deduce Theorem 5.2 from Lemma 5.9, we need to know that $\mathbb{P}(G_{n_0})$ is large for large n_0 provided (5.7) holds. This is the purpose of the next lemma, which plays a role similar to that of Lemma 4.4 in the proof of Theorem 1.7.

Lemma 5.10. *Suppose our parameters are such that (5.7) from Theorem 5.2 holds. Then for $n \in \mathbb{N}$,*

$$\mathbb{P}(G_n) = 1 - o_n^\infty(n).$$

Proof. Let $q_* := \frac{\alpha}{\alpha-1}$. By (5.7),

$$\frac{(s-1)(q_*-1)}{\bar{\kappa}_s q_*^\alpha} > \frac{1}{\bar{\kappa}_s}. \quad (5.25)$$

Hence assertion 2 of Lemma 5.6 implies that we can find a $\zeta > 0$ and a $\delta > 0$ such that with $D_{q_*+\zeta}$ as in (5.13),

$$D_{q_*(1+\zeta)}(z, q_*(\mathcal{Q}_s \setminus (P_{\mathbf{x}} \cup P_{-\mathbf{x}}))) \geq \frac{1 - q_*^{1-\alpha}}{\bar{\kappa}_s(\alpha - 1)} + \frac{1}{\underline{\kappa}_s} + \delta, \quad \forall z \in \partial \mathcal{Q}_s. \quad (5.26)$$

Now suppose given $n \in \mathbb{N}$ and set $n' := \lfloor q_*^{-1}n \rfloor$. By (5.20) of Lemma 5.7, it holds except on an event of conditional probability $1 - o_n^\infty(n)$ given $\mathcal{F}_{\tau_{n'}}$ that

$$\tau_{\bar{n}} - \tau_n \leq \frac{q_*^{\alpha-1}n^{1-\alpha} - n^{1-\alpha} + n^{1-\alpha-\beta}}{\bar{\kappa}_s(\alpha - 1)} + \frac{q_*^{\alpha-1}n^{1-\alpha}}{\underline{\kappa}_s}. \quad (5.27)$$

By (5.26) and the scaling property of D ,

$$\begin{aligned} & D_{(1+\zeta)n} \left(n(\partial \mathcal{Q}_s \setminus (P_{\mathbf{x}} \cup P_{-\mathbf{x}})), \mathcal{V}(A_{\tau_{n'}}) \cup B_{\zeta n'}^{\nu_s}(0) \right) \\ & \geq \frac{q_*^{\alpha-1}n^{1-\alpha} - n^{1-\alpha}}{\bar{\kappa}_s(\alpha - 1)} + \frac{q_*^{\alpha-1}n^{1-\alpha}}{\underline{\kappa}_s} + q_*^{\alpha-1}n^{1-\alpha}\delta + O_n(n^{-\alpha}). \end{aligned} \quad (5.28)$$

The right side of (5.28) minus the right side of (5.27) is $\geq n^{1-\alpha}$ for large enough n . By Lemma 5.8, we infer that except on an event of conditional probability $1 - o_n^\infty(n)$ given $\mathcal{F}_{\tau_{n'}}$, the vertex u_n lies at D -distance $\geq n^{1-\alpha}$ (and hence Euclidean distance $\geq n$) from $n(\partial \mathcal{Q}_s \setminus (P_{\mathbf{x}} \cup P_{-\mathbf{x}}))$. Therefore $u_n/\nu_s(u_n) \in P_{\mathbf{x}} \cup P_{-\mathbf{x}}$, i.e. G_n occurs. \square

5.4 Proof of Theorem 5.2

For $n_0 \in \mathbb{N}$, let E_{n_0} be the event that the three conditions of Lemma 5.9 are satisfied so that with G_{n_0} as in (5.17),

$$\mathbb{P}(E_{n_0} \mid \mathcal{F}_{\tau_{n_0}}) \mathbb{1}_{G_{n_0}} = (1 - o_{n_0}^\infty(n_0)) \mathbb{1}_{G_{n_0}}.$$

We always have $\mathbb{P}(G_{n_0}) > 0$ for each $n_0 \in \mathbb{N}$, so for large enough $n_0 \in \mathbb{N}$ we have $\mathbb{P}(E_{n_0}) > 0$. Furthermore, by Lemma 5.10 and the Borel-Cantelli lemma, whenever (5.7) holds a.s. G_{n_0} occurs for large enough $n_0 \in \mathbb{N}$. It therefore suffices to prove that if (5.5) and E_{n_0} occurs, then a.s. (5.8) holds (note that we use symmetry between \mathcal{K} and $-\mathcal{K}$ to obtain (5.6) in the case that (5.7) does not hold).

To this end, we first observe that (5.5) implies that there exists $q_0 > 1$ such that for each $q \geq q_0$,

$$\frac{1 + q^{1-\alpha} - 2^\alpha(q + 1 + s(q - 1))^{1-\alpha}}{\bar{\kappa}_s(\alpha - 1)} > \frac{1}{\bar{\kappa}_s(\alpha - 1)}. \quad (5.29)$$

By Lemma 1.6 and assertion 3 of Lemma 5.6, there is an $n_* \in \mathbb{N}$ such that whenever E_{n_0} occurs, $n \geq n_0 \vee n_*$, $q \in [q_0, 2q_0]$, and $v \in \mathbb{Z}^d$ with $\nu_s(v) = qn$ and $v/\nu_s(v) \in \partial \mathcal{Q}_s \setminus (P_{\mathbf{x}} \cup P_{-\mathbf{x}})$,

$$D\left(v, A_{\tau_n} \cup B_{q^{-1}n}^{\nu_s}(0)\right) \geq D(v, B_n^{\nu_s}(0)) + O_n(n^{-\alpha}) \geq \frac{n^{1-\alpha}}{\bar{\kappa}_s(\alpha - 1)} + 2n^{1-\alpha-\beta}.$$

By condition 3 of Lemma 5.9, it follows that if E_{n_0} occurs, then no such v belongs to $\mathcal{V}(A_{\tau_\infty})$. Hence if E_{n_0} occurs, then a.s. $\#(\mathcal{V}(A_{\tau_\infty}) \setminus (\mathcal{K} \cup (-\mathcal{K}))) < \infty$ so since \mathcal{K} is convex, a.s.

$$\#\{e \in \mathcal{E}(A_{\tau_\infty}) : e \not\subset \mathcal{K} \cup (-\mathcal{K})\} < \infty. \quad (5.30)$$

We will now apply Proposition 2.7 to show that on the event (5.30), a.s. either all but finitely many edges of A_{τ_∞} are contained in \mathcal{K} or all but finitely many edges of A_{τ_∞} are contained in $-\mathcal{K}$. Let $C > 0$ be chosen so that the graph distance from $\mathbb{Z}^d \cap (\mathcal{K} \setminus B_C^{|\cdot|}(0))$ to $\mathbb{Z}^d \cap (-\mathcal{K} \cap B_C^{|\cdot|}(0))$ is at least 6. Let Γ_1 (resp. Γ_2) be the largest subgraph of \mathbb{Z}^d which is contained in $\mathcal{K} \setminus B_C^{|\cdot|}(0)$ (resp. $-\mathcal{K} \setminus B_C^{|\cdot|}(0)$). Note that the graph distance in \mathbb{Z}^d between Γ_1 and Γ_2 is at least 3 and that (5.30) implies

$$\#(\mathcal{E}(A_{\tau_\infty}) \setminus \mathcal{E}(\Gamma_1 \cup \Gamma_2)) < \infty. \quad (5.31)$$

For $k \in \mathbb{N}$, let t^k be the smallest $t > 0$ for which $\#\mathcal{E}(A_t) = k$. Almost surely, there is a $k_* \in \mathbb{N}$ for which no edge of $\mathcal{E}(A_{\tau_\infty}) \setminus \mathcal{E}(A_{t^{k_*}})$ intersects $B_C^{|\cdot|}(0)$. If $\mathcal{E}(A_{\tau_\infty}) \setminus \mathcal{E}(A_{t^{k_*}})$ is disjoint from either $\mathcal{E}(\Gamma_1)$ or $\mathcal{E}(\Gamma_2)$, then we are done. Otherwise, Proposition 2.7 applied with $\tau = t^k$ for generic $k \in \mathbb{N}$ implies that whenever (5.31) holds, a.s. either $\mathcal{E}(A_{\tau_\infty}) \setminus \mathcal{E}(\Gamma_1)$ or $\mathcal{E}(A_{\tau_\infty}) \setminus \mathcal{E}(\Gamma_2)$ is finite, whence (5.8) holds. \square

6 Open problems

Here we list some open problems related to the model considered in this paper. We expect that the solutions to some of these problems may require additional knowledge of the Eden model limiting shape \mathbb{A} .

1. Under what conditions on f is the limit shape $\mathbb{B} = B_1^D(0)$ in Theorem 1.7 convex? Simulations suggest that this is not always the case when f is the α th power of a norm; see Figure 2.
2. It is easy to see from Lemma 1.6 that for any choice of f , the limit shape \mathbb{B} is compact, contains a neighborhood of 0, and that $\partial\mathbb{B}$ intersects each ray emanating from 0 exactly once. If $K \subset \mathbb{R}^d$ satisfies these three conditions and has Lipschitz boundary, does there exist an $\alpha < 1$ and an α -weight function f for which $K = \mathbb{B}$? If not, what conditions on K do ensure the existence of such an f ?
3. Does there exist an $\alpha = 1$ -weight function f such that the sets A_t a.s. converge to a limit shape in the sense of Theorem 1.7? What if we instead consider convergence of the re-scaled clusters A_t in the Hausdorff distance (which is a weaker mode of convergence than the one in Theorem 1.7)? What if we allow a random limit shape and relax a.s. convergence to convergence in law? We refer to the right panel of Figure 5 for a simulation in the case $\alpha = 1$.
4. Give a more general characterization than the one provided in Theorem 5.2 of the set of α -weight functions f for which a.s. all but finitely many vertices of A_{τ_∞} are contained in a cone of opening angle $< \pi$. Simulations suggest that this statement is true in much greater generality than the setting of Theorem 5.2; see, e.g., Figure 3.
5. If f is such that a.s. all but finitely many vertices of A_{τ_∞} are contained in a cone, what can be said about the law of the opening angle of this cone (as a function of f)?
6. What can be said about the model of Definition 1.3 if instead of exponential edge passage times X_e with parameter $\text{wt}(e)$, we consider a fixed random variable X and take the random variables X_e to be independent each with the law of $\text{wt}(e)^{-1}X$? Note that the results of [Kes93] do not require exponential passage times, but the proofs in the present paper use the Markov property (Lemma 2.1) which only works for exponential passage times.

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